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DISTRIBUTION OF INCOME AND WEALTH AMONG INDIVIDUALS[1]

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Implications for the distribution of wealth and income of alternative assumptions about savings, reproduction, inheritance policies, and labor homogeneity are investigated in the context of a neoclassical growth model. The paper isolates the different economic forces which tend to make the distribution of wealth in the long run equalitarian and those which tend to make wealth unevenly distributed.

1. INTRODUCTION

ALTHOUGH THE RECENT literature has abounded with alternative theories of distribution of income among factors of production, there have been few attempts to develop a theory of distribution of wealth and income among individuals.[2] The purpose of this study is to isolate some of the economic forces which in the long run tend to equalize wealth and some which tend to make it less evenly distributed. In particular, we examine the implications for distribution of alternative assumptions about the form of the consumption function, the heterogeneity of labor skills, inheritance policies, and the response of the reproduction rate to different levels of income.

We begin by considering a simple model of accumulation, with a linear savings function, a constant reproduction rate, homogeneous labor, and equal division of wealth among one's heirs. In such an economy, if the balanced growth path is stable, all wealth and income is asymptotically evenly distributed, with the possible exception, in the case of negative savings at zero income, of a group with zero wealth. In the process of accumulation, there may, however, be a period during which wealth becomes less evenly distributed. We then show that the basic conclusions are unaltered under a variety of alternative savings assumptions, where savings is a function of wealth or of the distribution of income, or where savings is a nonlinear concave function of income, and that variable rates of reproduction make no difference at least to the asymptotic results. The effects of alternative taxes on the speed of equalization are investigated in Section 4, and in Section 5 we consider a simple example to see the order of magnitudes of time that are involved in the equalization process.

In the remaining sections of the paper, we investigate the "forces for inequality": heterogeneity of labor force, class savings behavior, and alternative inheritance policies.

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[2] Notable exceptions to this are the works of Champernowne and Mandelbrot [**1, 5**], but their work suffers from the deficiency that the distribution of income is determined by a stochastic process, the character of which seems to have little to do with economic processes themselves. They seem to have little to say about, for instance, the relationship of the distribution of income among factors to the distribution of income among individuals.

PART I. THE FORCES FOR EQUALITY

2. THE BASIC MODEL

In this section (and throughout the paper), it will be convenient to think of society as divided into a number of groups; all the members of any one group have the same wealth but groups differ in their per capita wealth holdings.

We assume that labor is homogeneous (all workers receive the same wage). Thus, all the members of any one group have the same income as well as the same wealth. Each factor is paid its marginal product. We assume a concave, constant returns to scale production function;[3] if y is output per man and k is the aggregate capital-labor ratio, then

$$(2.1)\qquad y = f(k), \quad f'(k) > 0, \quad f'' < 0.$$

If w is the wage rate, and r the interest rate,[4]

$$(2.2)\qquad r = f'(k), \quad w = f(k) - kf'(k).$$

If y_i is the income per capita of group i, and c_i is the capital per man, then

$$(2.3)\qquad y_i = w + rc_i.$$

Savings per capita is assumed to be a linear function of income per capita. Hence if s_i is the per capita savings of group i, m the (constant) marginal propensity to save, and b the per capita savings at zero income, then

$$(2.4)\qquad s_i = my_i + b.$$

Reproduction occurs at a constant rate n, there is no intermarriage between income groups, and wealth is divided equally among one's offspring. These assumptions ensure that the proportion of the population in each group, a_i, remains constant.

We can now write down the basic equation of per capita wealth accumulation for group i:

$$(2.5)\qquad \frac{\dot{c}_i}{c_i} = \frac{s_i}{c_i} - n = \frac{b + mw}{c_i} + mr - n.$$

Moreover, if we let K_i be the total wealth holdings of group i, and define

$$(2.6)\qquad k_i = K_i/L = a_i c_i,$$

it is clear that

$$(2.7)\qquad k = \sum k_i = \sum a_i c_i.$$

Then the differential equation for aggregate capital accumulation is

$$(2.8)\qquad \dot{k} = \sum \dot{k}_i = \sum a_i \dot{c}_i = b + mw + mrk - nk.$$

[3] Satisfying the Inada derivative conditions.

[4] For most of the analysis, all we require is that the interest rate be a declining function of the capital-labor ratio and that the wage rate be an increasing function of the capital-labor ratio.

Observe that *the aggregate capital accumulation behavior is independent of the distribution of wealth.* This is an essential result of the linearity assumption.

In analyzing this model, we shall first discuss the aggregate balanced growth paths and their stability. We shall then discuss the conditions under which a given group is in equilibrium, i.e., has unchanging per capita wealth. Next, we will consider short and long run movements in the wealth distribution. Finally, we will investigate what these results imply for movements in the distribution of income.

Aggregate Balanced Growth Paths

If the economy is in balanced growth, $\dot{k} = 0$, i.e.,

$$my = nk - b. \tag{2.9}$$

In the case of $b = 0$, a strictly proportional savings function, this is simply the "Solow" equilibrium. If $b > 0$, there is a unique value for which $my = nk$, i.e., a unique aggregate balanced growth path. If, on the other hand, $b < 0$ (at a zero income a negative amount is saved), then there will in general exist two balanced growth paths.[5]

If there is only one balanced growth path, it is globally stable, since for capital-labor ratios greater than that of the balanced growth path, savings per capita is less than that required to maintain the same capital-output ratio with population growing at the rate n. The converse follows for capital-labor ratios less than that of the balanced growth path.

On the other hand, if there are two balanced growth paths (Figure 1), the lower one will be locally unstable and the upper will be locally stable. Differentiating the capital accumulation equation (2.8) with respect to k and evaluating it at $\dot{k} = 0$, we obtain

$$\frac{\partial \dot{k}}{\partial k} = mr - n. \tag{2.10}$$

The balanced growth path is stable or unstable as $\partial \dot{k}/\partial k$ is less than or greater than zero. The slope of the my curve is mr, and n is the slope of the $nk - b$ curve. Since my is concave, it is clear that the lower intersection must have $mr > n$ and the upper intersection must have $mr < n$.[6]

Thus, to the left of the lower equilibrium, savings per man is less than that required to sustain that capital-labor ratio, and hence the capital-labor ratio falls (continually).[7] Above the lower equilibrium, but below the upper equilibrium

[5] These results are contingent on the concavity of the production function and on the production function satisfying the Inada conditions.

[6] In the singular case of a tangency between the my curve and the $nk - b$ curve, where the upper and lower equilibria merge together, we have a stable-unstable equilibrium: stable with respect to upward deviations, unstable with respect to downward deviations. In this equilibrium, $mr = n$, the rate of profit is equal to the rate of growth divided by the marginal propensity to save.

[7] What happens when $k = 0$ is a question which we shall postpone for the moment. Negative k is possible only if there exist foreign countries from whom one can borrow. For a long run savings function it may well be argued on the basis of econometric evidence that b is zero; we prefer, however, to keep the analysis as general as possible.

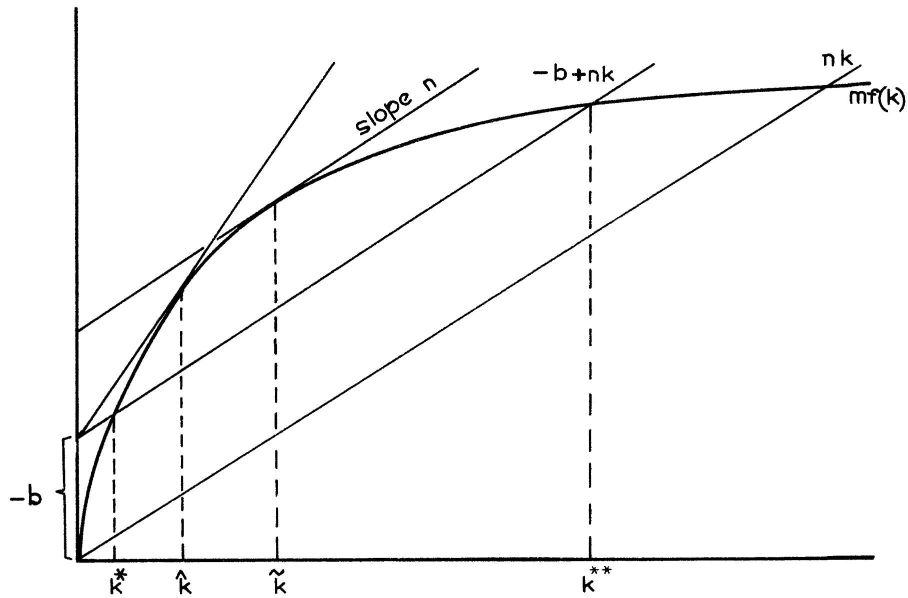

FIGURE 1

(between k^* and k^{**} on Figure 1) the reverse situation holds, so that the economy has an expanding capital-labor ratio. Finally, above the upper equilibrium (k^{**}), the economy has a declining capital-labor ratio.

Equilibrium for Wealth-Income Groups

Having analyzed the aggregate properties of the model, we turn now to investigate the behavior of the separate wealth-income groups.

It should be clear that for any given aggregate capital-labor ratio k, there can exist at most only one group, with per capita wealth c^*, which is in equilibrium, i.e., only one group whose per capita wealth is neither increasing nor decreasing. We require $\dot{c}_i/c_i = 0$ or

$$c^* = \frac{b + mw(k)}{n - mr(k)}. \tag{2.11}$$

(Observe that c^* is a function of k.)

If we define (see Figure 1)

$$w(\hat{k}) = f(\hat{k}) - \hat{k}f'(\hat{k}) = -b/m \tag{2.12}$$

and

$$r(\tilde{k}) = f'(\tilde{k}) = n/m, \tag{2.13}$$

then, because of concavity of $f(k)$, $k^* < \hat{k} < \tilde{k} < k^{**}$, and if

$$
\begin{array}{lllll}
 & k < k^*, & my + b - nk < 0, & mr - n > 0, & mw + b < 0; \\
 & k^* < k < \hat{k}, & my + b - nk > 0, & mr - n > 0, & mw + b < 0; \\
(2.14) & \hat{k} < k < \tilde{k}, & my + b - nk > 0, & mr - n > 0, & mw + b > 0; \\
 & \tilde{k} < k < k^{**}, & my + b - nk > 0, & mr - n < 0, & mw + b > 0; \\
 & k^{**} < k, & my + b - nk < 0, & mr - n < 0, & mw + b > 0.
\end{array}
$$

It immediately follows that for any given k, there is a unique wealth-income group in equilibrium, and that $c^* > 0$ if $k < \hat{k}$ or if $k > \tilde{k}$, but $c^* < 0$ if $\hat{k} < k < \tilde{k}$.

It should be observed, however, that in the first case, with $k < \hat{k}$, groups with per capita wealth less than c^* have decreasing per capita wealth, while if $k > \tilde{k}$, groups with per capita wealth less than c^* have increasing per capita wealth. (The converse is true for those with per capita wealth greater than c^*.) In the intermediate case, all groups with wealth greater than c^*, and in particular, all groups with positive wealth holdings, have increasing per capita wealth.

Movements in Distribution of Wealth

We examine now how the distribution of wealth changes over time. Without loss of generality, we consider the case of two income groups. We wish to know whether c_1 is growing faster or slower than c_2, given that $c_1 < c_2$. If it is growing faster, then the ownership of wealth (at least in a relative sense) is becoming more "equalitarian;" if it is growing slower, it is becoming less "equalitarian." But

$$(2.15) \qquad \dot{c}_1/c_1 - \dot{c}_2/c_2 = (b + mw)(1/c_1 - 1/c_2).$$

Hence, if $b + mw > 0$, the ownership of wealth becomes (relatively) more equalitarian, while if $b + mw < 0$, it becomes "worse." If $b = -mw$, there is no change in the (relative) ownership of property. Hence, to the left of $\hat{k}$ the ownership of capital is becoming more uneven; to the right it is becoming more even.

The economic reasoning behind this result should be clear. If $b + mw$ is equal to zero, increasing per capita wealth by a given percentage increases savings (mrc_i) by the same percentage, but it also increases the savings required to sustain that per capita wealth (nc_i) by the same percentage, so that whatever c_i happens to be, there it remains. If $b + mw$ is positive, increasing per capita wealth by a given percentage increases (per capita) savings by a smaller percentage, while the savings required to sustain that per capita wealth ratio goes up in proportion to c_i, and conversely for $b + mw$ less than zero.

Thus if the economy asymptotically converges to the upper equilibrium in the long run, there must be an equalitarian distribution of wealth, since at the upper equilibrium the wealth per man of the poorer groups grows faster than that of the richer groups. Indeed, if the economy is in balanced growth we can rewrite equation (2.15) as

$$(2.16) \qquad \frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j} = (n - mr)k\left(\frac{1}{c_i} - \frac{1}{c_j}\right).$$

Since the condition for stability for aggregate equilibrium is $n - mr > 0$, we have the general proposition that, *if the economy is at a stable equilibrium, the distribution of wealth must eventually be equalitarian.*[8]

But at the lower equilibrium, those groups with initial per capita wealth less than the equilibrium will grow continually poorer, while those groups with initial per capita wealth greater than the equilibrium will grow continually richer. This follows from

$$\begin{aligned} \dot{c}_i &= mw + b + mrk^* + m(c_i - k^*)r - nk^* - n(c_i - k^*) \\ &= (c_i - k^*)(mr - n) \gtrless 0 \qquad \text{as} \quad c_i \gtrless k^*. \end{aligned}$$

(And of course, those with more initial per capita wealth have a faster rate of growth of per capita wealth.)[9]

Thus it should be clear that although the fact that each of the individual groups is in equilibrium implies that the aggregate is in equilibrium, the converse is not true. The aggregate can be in equilibrium while the distribution of wealth is changing.

We are finally ready to fully describe movements of the distribution of wealth in our economy.

(i) There exist in general two balanced growth paths,[10] along which the capital-labor ratio, output-capital ratio, wage rate, etc., are all constant.

(ii) The one corresponding to the higher capital-labor ratio is stable both with respect to the aggregate (locally) and with respect to the component income classes (globally). If the overall capital-labor ratio is increased or decreased (provided it does not fall below k^*), the economy returns to the balanced growth path, and

[8] Thus, the fact that poor individuals have a lower average propensity to save than that of rich individuals does not necessarily mean that over time the distribution of wealth will become more disparate. See Friedman [**2**], especially Chapter IV.

[9] If there is a lower bound on the amount of capital that one can hold (an upper bound on indebtedness), then we must modify our savings functions. Assume that the lower bound is zero. Then

$$\begin{aligned} s_i &= b + mw + mrc_i, \quad & c_i > 0, \\ s_i &= 0, & c_i = 0. \end{aligned}$$

We assume that there are two groups, a "poor" group with zero wealth and with a of the population, and a "rich" group with $1 - a$ of the population and all the capital. Then

$$\begin{aligned} \dot{k} &= (1 - a)(b + mw) + (mr - n)k \\ &= (1 - a)b + mf(k)(1 - a\alpha(k)) - nk = \Psi(k) \end{aligned}$$

where $y = w + rk = f(k)$ and $\alpha(k) = w/f(k)$. Then $\Psi'(k) = mf' + kf''am - n$. Hence, for $k > \tilde{k}$, $\Psi'(k) < 0$, and thus there can exist at most one equilibrium with $k > \tilde{k}$.

But since $\Psi''(k) = mf''(1 + a) + f'''kma$, which depends on the third derivative of $f(k)$, there may exist more than one solution with $k < \tilde{k}$.

If the elasticity of substitution equals one, then there will exist at most two solutions, since the capital accumulation equation is identical to that examined earlier, with m replaced by $m(1 - a\alpha)$.

We can extend these results to the case where the lower limit of per capita wealth is not zero, but e. Then, in the balanced growth path, we can show that $k = ae + (1 - a)((b + mw)/(n - mr))$. We can show, as above, that there can exist at most one solution to this equation for $k > \tilde{k}$.

[10] There will be one if $b \geqslant 0$. If the Inada condition is not satisfied and the production function is not concave, then, of course, there may exist more or fewer equilibria.

if individual income classes are perturbed, the economy eventually returns to the equalitarian state.

(iii) The one corresponding to the lower capital-labor ratio is unstable, both with respect to the aggregate and with respect to the component income classes. If the aggregate k is decreased, it continues to decrease (forever); if it is increased, it continues to increase until it arrives at the upper equilibrium. If individual income classes are perturbed from the equal distribution position in such a way that the aggregate capital-labor ratio remains constant, the classes with per capita wealth greater than the overall capital-labor ratio continually increase their per capita wealth. The converse is true for those with less wealth than the "average."

(iv) If the economy is initially within the region between k^* and $\hat{k}$, then the overall capital-labor ratio is increasing and the economy eventually arrives in a state with completely equal distribution of income and wealth; but until the overall capital-labor ratio becomes equal to $\hat{k}$, the relative distribution of wealth becomes more uneven.[11]

(v) For all capital-labor ratios greater than $\hat{k}$, the distribution of wealth becomes (relatively) more even, eventually reaching complete equality.

Movements in the Distribution of Income

The adaptation of these results to movements in the distribution of income is straightforward. If the elasticity of substitution of the production function[12] is equal to one, then the analysis carries over exactly. If the elasticity of substitution is less than one, for instance, then

(a) in the region $\hat{k} < k < k^{**}$, the decreasing share of capital and the equalization of its ownership both serve to equalize the distribution of income;

(b) in the region $k > k^{**}$ the increasing share of capital and the equalization of its ownership offset each other; eventually, of course, the equalization tendencies dominate;

(c) in the region $k^* < k < \hat{k}$, the decreasing share of capital and the increasing spread in the ownership of capital offset each other; eventually, the economy moves into the region $\hat{k} < k < k^{**}$;

(d) in the region $k < k^*$, the increasing share of capital and the increasing spread in its ownership both serve to make the distribution of income more unequal.

The case of elasticity of substitution greater than unity may be analyzed similarly.[13]

[11] Perhaps one should not draw morals about the real world from such simple models. If the distribution of wealth appears in the short run to be becoming more uneven, do not lose hope in the capitalist system. Eventually (which may be a long time), the economy may lead to an equalitarian state, by its own accord.

[12] It should be noted that none of the results thus far has depended on the shape (except that f is concave and satisfies the Inada conditions) of the production function.

[13] So far we have assumed that there is no technological change. Hence, in the long run, all incomes (in steady state paths) are constant. If there is technological change, in the case of nonproportional linear savings hypothesis, we must make some adjustments in the analysis. But the basic qualitative results will, of course, be unaffected by Harrod neutral technical change.

3. ALTERNATIVE SAVINGS AND REPRODUCTION ASSUMPTIONS

The question naturally arises as to what extent the results obtained in the previous section depend on the particular assumptions made there. In this section, we show that the basic presumption for equalization obtains under a wide variety of specifications of savings and reproduction behavior.

Nonlinear Savings Functions

Let savings per capita of the ith group be a nonlinear function of income per capita of the ith group, $s(y_i)$. Then, for any given aggregate capital-labor ratio, k, there may be any number of income classes which are in equilibrium, i.e., for which $s(y_i) = nc_i$. But if the savings function is convex or concave there can be at most two groups, since

$$(3.1) \qquad \frac{d^2[s(y_i) - nc_i]}{dc_i^2} = s''r \gtrless 0 \quad \text{as} \quad s'' \gtrless 0.$$

But while aggregate savings behavior is independent of the distribution of income when the savings function is linear, it is not when the savings function is nonlinear. In general there will be any number of balanced growth paths, i.e., capital-labor ratios for which

$$(3.2) \qquad s(w(k) + r(k)c_i) = nc_i$$

where $k = \Sigma a_i c_i$. But if the savings function is concave, and the proportion of the population in each of the income groups is fixed, then there can be at most three balanced growth paths. Two will have only one class present and one will have two classes present.[14]

The two one-class equilibria have exactly the same stability properties as in the linear case, and nothing more need be said about it here. The two-class equilibrium has, as one might expect, properties of both the lower and upper equilibrium one-class economies; if part or the entire lower class is disturbed, so that their wealth per capita is less than that in equilibrium, they become increasingly poorer and if they become slightly richer (in per capita wealth terms) than in equilibrium, they become increasingly richer, until they "merge" with the upper class. Of course, we have been assuming throughout this process that as individuals shift their class membership the aggregate capital-labor ratio changes in the appropriate way. As a larger proportion of the population join the upper

[14] The one class cases require $c_i = k$. Hence we require $s(w + rk) = nk$. But since $s(k)$ is a concave function of k, and nk is a linear function of k, there can be at most two solutions.

In the two-class case, let a per cent of the population be in the lower equilibrium and $(1 - a)$ in the higher. Let $\bar{k} = ac_1(k) + (1 - a)c_2(k)$ where $c_1(k), c_2(k)$ are the solutions to equation (3.2) for given k. Then $dc_i/dk = -s'(c_i - k)f''/s'r - n$. For the lower class, $c_i < k$, if the savings function is concave, $s'r > n$. The converse holds for the upper equilibrium. Hence $dc_i/dk < 0$ and $d\bar{k}/dk < 0$. There exists at most one k for which $\bar{k} = k$. As in the linear case, there is of course one further possibility existing (provided that at "very large incomes" savings becomes approximately proportional to income)—the poor could reduce their capital to a lower bound of say zero while the rich become increasingly richer.

class, the aggregate capital-labor ratio must rise. But as it rises, it leads all the other members of the lower class out of equilibrium, and since the lower equilibrium is unstable, there is no mechanism for them to reach equilibrium. It is unlikely then that any two-class equilibrium situation could ever be maintained for long.

Hence, in this model as in the linear model first examined, if the balanced growth path is stable, there is a tendency in the long run for the equalization of wealth and income—with the possible exception of a group (in an underdeveloped economy perhaps almost the entire economy) whose wealth is at some lower bound (zero, or the upper bound on indebtedness).

Savings as a Function of Wealth and Income

Recent investigations into savings functions have indicated that savings may be a function of wealth as well as income, e.g., $s = b + my + zc$ so

$$\dot{c}_i/c_i = \frac{(b + mw)}{c_i} + mr + z - n. \tag{3.3}$$

The analysis proceeds exactly as in Section 2 of this paper, and (3.3) is identical to (2.5) with n replaced by $n - z$. If z is positive, then it is as if the rate of population growth is smaller than it actually is, so that the equilibrium capital-labor ratio is higher, r is lower, w is higher, etc. The more reasonable assumption is to make z negative, indicating that the more wealth one has, for any given income, the less one saves (as for instance some of the life cycle stories suggest); then it is as if n is higher, i.e., the equilibrium capital-labor ratio will be lower, wages will be lower, and the profit rate will be higher.

An alternative formulation of savings behavior is the following: individuals have a desired wealth-income ratio, given by q^*. If the wealth-income ratio is less than the desired, they accumulate. If it is greater than the desired, they decumulate. We may write the adjustment process as

$$\dot{c} = h(c^* - c), \tag{3.4}$$

where

$$c^* = q^*y = q^*(w + rc). \tag{3.5}$$

Substituting, we have

$$\dot{c} = h[q^*(w + rc) - c] = hq^*w + (q^*rh - h)c. \tag{3.6}$$

Since $k = \Sigma ac_i$,

$$\dot{k} = hq^*w + (rq^*h - h)k = hq^*y - hk. \tag{3.7}$$

There is a unique balanced growth path, with $q^* = y/k$, and it is stable. Moreover, for any given aggregate capital-labor ratio, there is at most one c for which $\dot{c} = 0$:

$$c = \frac{q^*w}{1 - rq^*}. \tag{3.8}$$

This is meaningful only if $r(k) < 1/q^*$, i.e., for very low capital-labor ratios there exists no positive c for which $\dot{c} = 0$. In all cases, however, the poor accumulate capital faster than the rich, since

$$\dot{c}_1/c_1 - \dot{c}_2/c_2 = hq^*w(1/c_1 - 1/c_2), \tag{3.9}$$

and eventually all wealth is evenly distributed.

Classical Savings Function

Another savings hypothesis which has been strongly advocated, particularly in Cambridge, is the classical or Kaldorian savings function [**3**] where different proportions of profits and wage income are saved. Again, because of the linearity assumption, the behavior of the economy as a whole is unaffected by the distribution of wealth and income. Except when $s_w = 0$, asymptotically, all wealth is evenly distributed:

$$\frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j} = (s_w w)\left(\frac{1}{c_i} - \frac{1}{c_j}\right),$$

where s_w is the savings propensity out of wages. In the singular case where $s_w = 0$, there is no tendency for equalization. Indeed, in balanced growth, since the aggregate capital-labor ratio is fixed, increases in wealth by one group must come at the expense of others.

Variable Rates of Reproduction

In this subsection, we assume that the rate of reproduction of the ith group is a function of its per capita income $n_i = n(y_i)$. For simplicity, we shall revert to the linear savings assumption. It is clear that different groups will in general have different rates of reproduction and the group with the highest rate of reproduction "dominates" the entire population. All groups except the dominant one asymptotically "disappear." Assume there are only two groups—the rich and the poor. If the rich reproduce more quickly than the poor, then although it is true that "the poor have ye always among you," in a relative sense they disappear. On the other hand, even if the rich reproduce more slowly than the poor, so that they become an infinitesimal part of the population, they may still have more than an infinitesimal part of the wealth of the total economy. To see this, if K_r is the capital of the rich and K_p that of the poor, and a is the proportion of the rich (asymptotically $a = 0$), then

$$\frac{d\ln(K_r/K_p)}{dt} = \frac{\dot{a}}{a(1-a)} + \frac{\dot{c}_r}{c_r} - \frac{\dot{c}_p}{c_p} = (mw + b)\left(\frac{1}{c_r} - \frac{1}{c_p}\right) + (n_r - n_p).$$

By assumption, $n_r < n_p$. If the economy is in an unstable equilibrium, $b + mw$ is negative so the wealth of the rich may be growing faster than that of the poor.

But if the economy is in a stable equilibrium,

$$\frac{d \ln(K_r/K_p)}{dt} < 0$$

so that asymptotically, the rich are infinitesimal not only in numbers but also in total wealth holdings, relative to the total economy.

4. TAXATION AND EQUALIZATION

Taxes for redistribution do more than just redistribute income today—they increase the rate at which wealth is equalized. To see this, consider the effects of a proportional income tax, in which all the proceeds are divided equally among the citizens.

If group i's before tax income is $y_i = w + rc_i$, its after tax income is

$$y_i' = (w + rc_i)(1 - t) + t(w + rk)$$

and hence the per capita wealth accumulation behavior of the economy remains unchanged. The relative movements in per capita wealth of two groups are given by

$$(4.1) \qquad g_c = \frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j} = (b + mw)\left(\frac{1}{c_i} - \frac{1}{c_j}\right) + mrtk\left(\frac{1}{c_i} - \frac{1}{c_j}\right),$$

and the change in the speed of equalization from the no-tax situation is

$$(4.2) \qquad \Delta g_c = mrtk\left(\frac{1}{c_i} - \frac{1}{c_j}\right).$$

Again, we observe that for "high" k, the poor increase their per capita wealth relative to the rich, until incomes and wealth are completely equalized, while for "low" k the rich grow richer relative to the poor. But note the two effects of the income tax:

First, the critical k which determines whether there is wealth equalization or not is lower, since now the condition is not $b + mw = 0$, but $b + mw + mrtk = 0$. In fact, at a tax rate greater than $1 - (n/mr)$, even at the lower balanced growth path equalization of wealth will occur.

Second, the rate at which equalization occurs is increased (or, if the distribution becomes more uneven, it does so at a slower rate than in the absence of the tax).

Similarly, the effects of progressive income taxes, profits taxes, and wealth taxes may be analyzed. It can be shown that for taxes of the same revenue the redistributive effects of either a profits tax or a progressive income tax are greater than those of the proportional income tax.

5. THE SPEED OF EQUALIZATION: AN EXAMPLE

In this section we shall work through an example to give the rough orders of magnitudes of the time involved. We take the Cobb-Douglas production function,

$y = k^{\alpha}$. If $b = 0$, the differential equation for the aggregate capital-labor ratio is

$$\dot{k} = mk^{\alpha} - nk.$$

This can easily be solved explicitly for $k(t)$:

$$k(t) = \left\{\left(k(0)^{1-\alpha} - \frac{m}{n}\right)e^{(\alpha-1)nt} + \frac{m}{n}\right\}^{1/(1-\alpha)}$$

so that

$$\lim_{t\to\infty} k(t) = \left(\frac{m}{n}\right)^{1/(1-\alpha)} = k^*.$$

If we use as our definition of distance from equilibrium $V(k) = (k(t) - k^*)^2$, then the rate of change of $V(k)$ is given by

$$\frac{d \ln V(k)}{dt} = -2\frac{k(mk^{\alpha-1} - n)}{\{(k(0)^{1-\alpha} - m/n)\, e^{(\alpha-1)nt} + m/n\}^{1/(1-\alpha)} - (m/n)^{1/(1-\alpha)}}.$$

On the other hand, the differential equation for per capita wealth of group i is

$$\dot{c}_i = mw + (mr - n)c_i = mk(t)^{\alpha}(1 - \alpha) + (m\alpha k(t)^{\alpha-1} - n)c_i$$

so

$$c(t) = e^{-\int_0^t (n - m\alpha k(v)^{\alpha-1})dv}\left[c(0) + \int_0^t mk(\tau)^{\alpha}(1 - \alpha)\, e^{\int_0^\tau (n - m\alpha k(v)^{\alpha-1})dv}\, d\tau\right].$$

If the economy is in aggregate equilibrium, this takes on the simple form

$$c_i(t) = (c_i(0) - k^*)\, e^{(mr-n)t} + k^*$$

or

$$c_i(t) = (c_i(0) - k^*)\, e^{(\alpha-1)nt} + k^*.$$

If we use as our measure of inequality of wealth the variance

$$V(c) = \sum_i a_i(c_i(t) - k)^2,$$

we can immediately calculate the speed of equalization when the economy is in aggregate equilibrium:

$$\frac{d \ln V(c)}{dt} = 2(\alpha - 1)n.$$

It should be noted that the speed of equalization depends only on the rate of growth and the share of labor, and in fact is proportional to each.

To get an idea of the numerical values, let us assume that $\alpha = .25$, $n = .01$, and $m = .2$. Then the "half-life of wealth inequality" is 46.2 years, i.e., in 46.2 years, the variance in wealth is reduced in half. If, on the other hand, $k = .8k^*$, so $V(k) = .04k^*$, it takes 46.4 years for $V(k)$ to be reduced in half (i.e., for $k = .8586k^*$).

We have already noted that the speed of equalization is very sensitive to n and α. If, for instance, the rate of growth increases from one per cent to two per cent, the "half-life" is reduced from 46.2 to 23.1 years.

Part II

In the first part of this paper, we have identified some strong long term forces leading the economy to equalization of wealth and income. There are, on the other hand, several forces tending to preserve inequality in wealth and income. The forces that we shall focus on in particular are (a) heterogeneity of the labor force, (b) "class" type savings behavior, and (c) alternative inheritance policies.

6. Heterogeneous Labor Force

In this section we assume that some labor is more productive than other labor and receives accordingly a higher wage. We further assume that different kinds of labor are related to each other in a "pure labor augmenting way" so the ratio of the wage of any two groups is constant, and there is no intermarriage between groups. We revert to the assumption of a constant rate of growth of population and a linear savings function.

If p_i is the number of efficiency units incorporated in each member of group i, it is easy to show that in equilibrium c_i, per capita wealth of the ith group, is a linear function of p_i:

$$c_i = \frac{b + mp_i w}{n - rm}.$$

Thus, for any given distribution of productivities, we can derive the resulting asymptotic distribution of wealth. If $g(p)$ is the density function of p, then the density function of c is

$$\frac{n - rm}{mw} g\left(\frac{c(n - rm) - b}{mw}\right).$$

If productivities are normally distributed, wealth will be normally distributed; if productivities are lognormally distributed, wealth will also be distributed lognormally (but a three parameter lognormal function).

Further insight into this economy may be had if we assume that the economy has only two classes—an efficient class with $p = 1$ and an inefficient class with $p < 1$. The economy is on a stable balanced growth path; thus $n - mr > 0$ and $b + mw > 0$. Then, if p is sufficiently small, i.e., $p \leqslant -b/mw$, all of the capital will be owned by one class. In fact, if $p < -b/mw$, the poorer class actually goes into debt to the richer class, and there exists an equilibrium per capita debt of the poorer class. The rich save enough to lend to the poor and sustain the capital-labor ratio.

If $p < -b/mw$, but a constraint is imposed on borrowing (say, no borrowing is allowed at all), then we have a two-class economy in which the poor consume everything and the rich (the capitalists) consume a proportion of their income. Denoting the efficient group by a subscript one,

$$\dot{k} = a_1 \dot{c}_1 = a(mw + mrc_1 - nc_1 + b).$$

In balanced growth $\dot{k} = 0$, so

$$r = \frac{n - (a(b + mw)/k)}{m} = \frac{n}{m} - \frac{ab}{mk} - \frac{aw}{k}.$$

If aw/k, the wage income of the rich divided by the *total* capital stock, is small and $b = 0$, we see that $r \approx n/m$, *the rate of profit is equal* (approximately) *to the rate of growth divided by the propensity to save* (of the "rich"), exactly the result of the Cambridge theory of distribution [**7**].

7. CLASS SAVINGS BEHAVIOR

The presence of different classes in the economy with different savings behavior may also give rise to disparities in the distribution of wealth. Consider a two-class economy: a capitalist class which does not work and saves s_j of its profits, and a workers class which derives its income from wages plus return on the capital previously saved and saves s_i of its income (regardless of the source). Models with this savings behavior have been investigated by Pasinetti [**7**], Meade [**6**], Samuelson and Modigliani [**8**], and Stiglitz [**9**]. Because of the linear savings assumption, the aggregate capital accumulation behavior is independent of the distribution of wealth. Thus, there is at most one two-class balanced growth path, (i.e., a balanced growth path with both capitalists and workers present). Along this balanced growth path, $r = n/s_j$. It is easy to see that distribution of wealth among the capitalists is an historic accident, and, as in the Kaldorian case with savings out of wages equal to zero, increases in the capital of one capitalist occur at the expense of other capitalists. All "workers," on the other hand, have the same wealth and income asymptotically, since for any group,[15]

$$\begin{aligned} \dot{c} &= s_i w + s_i r k_w + s_i r(c - k_w) - nk_w - n(c - k_w) \\ &= (s_i r - n)(c - k_w), \end{aligned} \tag{7.1}$$

where k_w is the capital per man owned by workers. Since $s_i < s_j$, $s_i r - n < 0$, so that if any labor group has per capita wealth greater than the average, k_w, its per capita wealth declines. The converse holds for any labor group with less per capita wealth than the average.

There also exists a unique balanced growth path with only workers present (the "dual regime" of [**8**]). But this case is identical to that investigated above in Section 2 with $b = 0$, for which we have already shown that asymptotically all wealth is evenly distributed.

[15] Since in balanced growth, $\dot{k}_w = s_i w + s_i r k_w - nk_w = 0$.

8. PRIMOGENITURE

So far in this paper we have considered only cases where wealth was divided equally among one's children. Without going into a detailed exposition of alternative inheritance programs, let us consider the case perhaps most contrary to that which has been discussed thus far, that of primogeniture (all wealth being left to the first born son). To carry through the analysis we shall need to introduce some further simplifications, and shift the analysis to discrete time.

We consider a period in which the population doubles itself. Each "family" has exactly two sons and two daughters. For simplicity, we shall say that children are born at the end of the period. Everybody lives for only one period, parents dying after giving birth to their quadruplets. We shall examine only equilibrium paths. Then, at the beginning of any period one half of the population has zero capital. Of the remainder, one half are born to fathers who were first born, one half to fathers who were not. (1/2)(1/2) of the population has $b + mw$ wealth per capita. Of the remainder, one half are born to fathers who were first born and one half to fathers who were not, so (1/2)(1/2)(1/2) of the population has $b + mw + (1 + mr) \times (b + mw)$ wealth per capita. And so on.

If we number our groups from the poorest to the richest, then the ith group has $(1/2)^{i+1}$ of the population (where the 0th group has zero wealth) and has a per capita wealth of

$$\frac{(b + mw)[(1 + mr)^i - 1]}{(1 + mr) - 1} = \frac{b + mw}{mr}[(1 + mr)^i - 1] \qquad (i = 1, \ldots n).$$

If we compare any two groups, their ratio of per capita wealth is

$$\frac{(1 + mr)^i - 1}{(1 + mr)^j - 1}.$$

For large i this yields the distribution function of an asymptotically Pareto form:

$$g(c > c^*) \approx \gamma c^{*-\ln 2/\ln(1+mr)},$$

where $\ln \gamma = \ln \frac{1}{2}[1 - (\ln(b + mw/mr)/\ln(1 + mr))]$ and where $g(c > c^*)$ is the proportion of the population with per capita wealth greater than c^*. To see this, note that the proportion of the population in the ith group is always equal to the proportion in all groups whose index is greater than i. For large i, $(1 + mr)^i - 1$ is approximately $(1 + mr)^i$, so the proportion of the population whose wealth is greater than $(1 + mr)^i(b + mw)/mr$ is $1/2^{i+1}$, and the result is immediate. If $mr = .6$ (recall that r is the rate of return over a generation, e.g., if the rate per year is .05, a generation is thirty years, then $r \approx 4.48$), then the exponent of c is 1.48, an empirically reasonable value.[16]

[16] To derive the aggregate capital-labor ratio of the economy, k, observe that k is simply a weighted sum of c_i, where the weights are proportions in the population. If $\frac{1}{2}(1 + mr)$ is less than unity, the infinite sum converges to $k = b + mw/(1 - mr)$. If we rewrite this equation as $b + my = k$ and observe that $n = \Delta L/L = 1$ in our discrete model, we obtain exactly the Solow growth equation—aggregate equilibrium is unaffected.

9. OTHER SOURCES OF INEQUALITY

Among the more important sources of inequality not discussed here are the following:

(a) Life cycle savings. If individuals save over their life time in a manner suggested by the life cycle hypothesis, the age distribution will be one of the prime determinants of the wealth distribution. There is some evidence to support this (see, e.g. [**4**]).

(b) Stochastic elements. Throughout the above discussion, we have made the assumption, conventional in growth theory, that there is no randomness in the rate of return on capital, in the rate of reproduction, etc. We have already noted that the theories of income inequality of Champernowne and Mandlebrot [**1**, **5**] are based primarily on stochastic models. Champernowne and Mandlebrot have shown, for instance, that if: (i) $c(t)$ is a Markovian sequence in discrete time, (ii) for large c, $\log c(t + 1) - \log c(t)$ is a random variable independent of $c(t)$, (iii) for large c, $E(c(t + 1) - c(t))$ for given $c(t)$ is negative, and (iv) for small c, the transition probabilities are such that not all $c(t)$ can become zero, then c (for large c) would have the Pareto distribution. They unfortunately have not provided any economic justification for these assumptions. A slight modification of our model can, however, provide us with an economic motivation for them. If, for instance, we assumed that the rate of return on capital is a random variable, uncorrelated with the amount of wealth an individual owned (at least for large c) but with an average value equal to the marginal product of capital, then (in the discrete time analogue of our model of Section 2), all the Champernowne conditions are satisfied, provided only that the economy has a sufficiently large[17] capital-labor ratio.

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[17] In the terminology of Section 2, provided $k > \tilde{k}$.