SCHOOLS OF THOUGHT IN ECONOMICS 4

SRAFFIAN ECONOMICS

VOLUME II

Ian Steedman

Sraffian Economics
Volume II

Edited by

Ian Steedman

Professor of Economics
University of Manchester

1988
EDWARD ELGAR

Contents

Acknowledgements

The editor and publishers wish to thank the following who have kindly given permission for the use of copyright material.

Academic Press, Inc. (London) Ltd for articles: H. Gram (1985), 'Duality and Positive Profits', *Contributions to Political Economy*, 4, pp. 61–77; N. Salvadori and I. Steedman (1985), 'Cost Functions and Produced Means of Production: Duality and Capital Theory', *Contributions to Political Economy*, 4, pp. 79–90.

Franco Angeli Editore Riviste S.R.L. for article: I. Steedman (1980), 'Returns to Scale and the Switch in Methods of Production', *Studi economici*, XXXV, pp. 5–13.

Australian Economic Papers for article: G. Hodgson and I. Steedman (1977), 'Depreciation of Machines of Changing Efficiency: A Note', *Australian Economic Papers*, 16, pp. 141–7.

Basil Blackwell Ltd for articles: C. Filippini and L. Filippini (1982), 'Two Theorems on Joint Production', *Economic Journal*, 92, pp. 386–90; I. Steedman (1977), 'Basics, Non-basics and Joint Production', *Economic Journal*, 87, pp. 324–8; I. Steedman (1982), 'Joint Production and the Wage–Rent Frontier', *Economic Journal*, 92, pp. 377–85; D.M. Nuti (1970), 'Capitalism, Socialism and Steady Growth', *Economic Journal*, 80, pp. 32–57; A. Bhaduri (1969), 'On the Significance of Recent Controversies on Capital Theory: A Marxian View', *Economic Journal*, 79, pp. 532–39; I. Steedman (1975), 'Positive Profits with Negative Surplus Value', *Economic Journal*, 85, pp. 114–23.

L. Cappelli S.P.A. for articles: B. Schefold (1976), 'Reduction to Dated Quantities of Labour, Roundabout Processes, and Switches of Techniques in Fixed Capital Systems', *Metroeconomica*, 28, pp. 1–15; G. Montani (1975), 'Scarce Natural Resources and Income Distribution', *Metroeconomica*, 27, pp. 68–101; L.L. Pasinetti (1973), 'The Notion of Vertical Integration in Economic Analysis', *Metroeconomica*, 25, pp. 1–29.

Kyklos Verlag for article: L. Mainwaring (1974), 'A Neo-ricardian Analysis of International Trade', *Kyklos*, 27, pp. 537–53.

The Manchester School for articles: R. Franke (1984), 'Some Problems Concerning the Notion of Cost-minimizing Systems in the Framework of Joint Production', *Manchester School*, LII, pp. 298–307; N. Salvadori (1985), 'Switching in Methods of Production and Joint Production', *Manchester School*, LIII, pp. 156–78; I. Steedman (1972), 'Jevons's Theory of Capital and Interest', *Manchester School*, XL, pp. 31–52.

Oxford University Press for article: H. Kurz (1978), 'Rent Theory in a Multisectoral Model', *Oxford Economic Papers*, 30, pp. 16–37.

Political Economy for articles: B. Schefold (1985), 'Sraffa and Applied Economics: Joint Production', *Political Economy*, 1, pp. 17–40; I. Steedman (1985), 'Joint Production and Technical Progress', *Political Economy*, 1, pp. 41–52.

Scottish Economic Society for article: R.L. Meek (1961), 'Mr. Sraffa's Rehabilitation of Classical Economics', *Scottish Journal of Political Economy*, 8, pp. 119–36.

Société Suisse de Statistique et d'Economie Politique for articles: K.R. Bharadwaj (1970), 'On the Maximum Number of Switches between Two Production Systems', *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, CVI, pp. 409–29; E. Zaghini (1967), 'On Non-basic Commodities', *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, CIII, pp. 257–66.

Springer Verlag for articles: N. Salvadori (1982), 'Existence of Cost-minimizing Systems within the Sraffa Framework', *Zeitschrift für Nationalökonomie*, 42, pp. 281–98; B. Schefold (1976), 'Relative Prices as a Function of the Rate of Profit: A Mathematical Note', *Zeitschrift für Nationalökonomie*, 36, pp. 21–48.

Every effort has been made to trace all the copyright holders but if any have been inadvertently overlooked the publishers will be pleased to make the necessary arrangement at the first opportunity.

In addition the publishers wish to thank the Library of the London School of Economics and Political Science and the British Library Document Supply Centre for their assistance in obtaining these articles.

Introduction

The introduction to *Sraffian Economics, Volume I* set out some of the capital theory background to Sraffa's great *Production of Commodities by Means of Commodities* (1960), outlined those arguments of Sraffa's book which do not relate to joint production and, finally, presented a brief guide to the critique of the marginal theory of value and distribution which came to be built on those arguments. Since it would be otiose to repeat that material here, we shall now proceed directly to consider Part II of Sraffa's book, which deals with joint production and fixed capital. Subsequent developments in the Sraffian discussion of these matters will then be introduced, as will a number of other theoretical and historical strands of work which have been inspired by Sraffa's book.

1. 'Multiple-product industries and fixed capital'

Part I of *Production of Commodities* is concerned exclusively with systems of production in which no fixed capital is used, that is, all produced inputs are fully used up within one cycle of production, and in which each process of production produces only one output. Part II, therefore, introduces multi-product industries (that is, joint production) and, as an important special case of joint production, the use of fixed capital. In the first chapter of Part II, Chapter VII on 'Joint Production', Sraffa notes immediately that if one (or more) processes produce more than one commodity, then it is no longer immediately evident – by contrast with the single products case – that there will be as many processes in operation as there are commodities produced: hence there might not be as many price–cost equations as there are commodity prices to be determined. Invoking but not discussing at length the consideration that it might well be necessary to operate more than one process producing given types of commodity, in order that the pattern of output should match the proportions in which commodities 'are required for use' (p. 43, n. 2), Sraffa then *assumes* that the number of processes used does in fact equal the number of commodities produced. (The price–cost equation system will thus be 'square'.)

Chapters VIII and IX are then devoted to a number of complications introduced by the presence of joint production. In the former, it is noted that the 'Standard Commodity' may now include negative amounts of some commodities (pp. 47–8) and that, consequently, in deciding which mathematical solution of the Standard system is the *economically* relevant solution, one can no longer simply select that with a non-negative Standard Commodity (pp. 53–4). More importantly, perhaps, Sraffa explains why his Part I distinction between basic commodities and non-basic commodities can no longer be applied in such a simple way and proposes a more abstract definition of the distinction, using the algebraic concepts of linear dependence and independence (pp. 49–53). In Chapter IX it is noted that, in the presence of joint production, a 'square' economic system may impute a *negative* amount of labour to the (direct and indirect) production of one or more commodities. That is, in a terminology which Sraffa does not employ, with joint production one or more commodities may have a negative Marxian 'labour value' (p. 60). Moreover, the 'reduction to dated quantities of labour', which Sraffa employed in Chapter VI of Part I, is not generally possible in the presence of joint production (pp. 58–59). And one can no longer even be certain that the real wage, however measured, must always be inversely related to the rate of profits. It is now possible that there are standards of value (numéraires) in terms of which the real wage moves non-monotonically as the rate of profits is increased (pp. 61–2).

Sraffa's Chapter X is devoted to fixed capital, which he describes as being the 'leading species' of the 'genus' joint products (p. 63). The connection is made as follows. Following Torrens, Ricardo, Malthus and Marx (pp. 94–5), Sraffa considers a production process which includes, say, a new machine amongst its inputs and describes the outputs from that process as including a 'one-year-old' machine. A second process includes a 'one-year-old' machine amongst its inputs and a 'two-year-old' machine amongst its outputs, etc. Thus even if there are no other elements of ('pure') joint production present, the joint production schema can be used to represent the use of durable capital goods. And, in general, that schema should be used to represent fixed capital using systems, for only in this way can a proper theory of depreciation be developed (other than in special cases). Sraffa shows how the joint products approach includes as a special case but is far more general than the standard 'annuity formula' for calculating the annual charge on a machine of constant efficiency (pp. 64–7). He notes that while fixed capital systems are not amenable, in general, to the 'reduction to dated quantities of labour' analysis (pp. 67–8), they need not cause such great problems for the Standard Commodity as do general joint products systems (pp. 72–3). More interestingly, perhaps, he also shows in detail how the value of a machine varies with its age and with the rate of profits (pp. 68–70) and how the value of a complete set of machines of various ages must vary with the rate of profits, even relative to the value of a new machine (pp. 70–2).

The final chapter of Part II is Chapter XI, 'Land', in which Sraffa considers the theory of rent, both in the context of the use of different qualities of land and in that of the use of more than one method of production on a given quality of land. It is noted that the relative fertility of different qualities of land cannot be defined, in general, independently of the rate of profits (p. 75) and that the presence of land requires the reconsideration of the distinction between single product and multiple product systems (pp. 77–8).

We may conclude this section by remarking that in the final section of his book (section 96 of Chapter XII, the sole chapter in Part III), Sraffa seeks to extend his discussion of switches between alternative systems of single product processes (sections 92–95) to the case of switches between multiple product systems.

2. Joint production and fixed capital further considered

The initial response to Sraffa's *Production of Commodities* concentrated on his discussion of circulating capital, single product systems, for that discussion provided a sufficient basis for the reswitching and capital-reversing critique of the marginal theory of value and distribution. (See *Sraffian Economics, Volume I*). But it was important to consider Sraffa's joint production arguments as well, both because joint production is, empirically, very important both in its fixed capital form *and* in its 'pure' form (*cf*. Steedman, 1984) and because joint production issues have been used as the basis for strong criticisms of 'classical' economic theory, the very kind of theory to which Sraffa seems to suggest a return (e.g. Jevons, 1970 [1871], pp. 208–12). Part I of the present volume contains some of the papers which have contributed to the discussion of these issues. (See also Pasinetti (ed.), 1980; Bidard (ed.), 1984; Salvadori and Steedman, 1988, for further papers and a survey.) Analysis of the topics involved can become rather technical and it would be quite inappropriate to attempt here a full summary of the relevant literature; rather, a brief sketch of a few of the findings will be attempted.

An early contribution was that of Manara (1968). He examined the conditions which must be satisfied by the quantities of the inputs to and the outputs from a multiple product system, in order that a 'square' system of price–cost equations should have an economically meaningful solution; conditions which had not been explicitly stated in full by Sraffa in Part II of his book. Manara also pointed out that Sraffa's equations to determine the Standard Commodity and the maximum rate of profit might yield not only Standard Commodities with negative components (as Sraffa had noted) but even Standard Commodities with complex elements (as Sraffa had not apparently noticed). These possibilities, which do not arise in single product systems, naturally raise severe doubts as to the economic significance of the Standard Commodity and of the linear wage–profit frontier which is obtained when the wage is measured in terms of that standard. Finally, Manara presented a very clear linear algebra formulation of Sraffa's distinction between basic and non-basic commodities in the multiple products case. (Manara's formulation was later combined with Pasinetti's work on 'vertical integration' – see section 3 below – to provide a more intuitive interpretation of basics and non-basics in joint production systems; see Steedman (1977) in which, however, non-basics are more adequately characterized than are basics.)

For the purposes of his discussion, Manara followed Sraffa (see section 1 above) in *assuming* that the joint products system to be analysed is a 'square' system – and much other valuable work has, indeed, also been based on this assumption. It will be clear, nevertheless, that such an assumption must, in the end, either be rigorously justified or be abandoned. This issue is still the object of active research and discussion and here we can only note one special case of general joint production in which 'squareness' can be established as a conclusion, rather than being simply assumed. If a multiple products system is undergoing steady growth at a rate equal to the rate of profit, if the consumption vector is independent of prices and of distribution, if there is free disposal and if, finally, only products with a positive price are named 'commodities', then competitive forces will establish the result that the number of processes in use will equal the number of 'commodities'. (See Steedman, 1976; Schefold, 1978, 1980.) 'Squareness' results can also be obtained in some fixed capital systems involving no element of 'pure' joint production but it remains an open question how general is the acceptability of 'square' systems.

The analysis of choice of technique is, of course, central to the Sraffa-based critique of marginal theory and it has therefore been important to examine and to develop Sraffa's brief remarks on this question in the context of joint production systems, remarks which took up only the closing section of Sraffa's book (section 96). In the single products case, it can be shown that the choice of a cost-minimizing technique, at a given rate of profit, is equivalent to the choice of a wage rate maximizing technique, at that rate of profit. Consequently, the economically more fundamental notion that competition leads to a cost-minimizing choice of technique can conveniently be represented, in diagrammatic terms, by saying that competition always takes the economy to the outermost wage–profit frontier. In joint production systems, alas, the two criteria need not coincide, for the cost-minimizing technique is not necessarily the wage rate maximizing one; here one must work with the former, more fundamental criterion of choice and cannot resort to the more intuitive presentation in terms of always reaching the outermost wage–profit frontier. Further complexities introduced into the choice of technique analysis by the presence of joint products include the fact that, in this more general case, there need not even *be* a cost-minimizing technique. It is possible, that is, that at the prices of system A it would be profitable to change to system B, while at the prices of system B it would be profitable to change to system A! (*Cf.* Bidard, 1984; Salvadori, 1982, 1984, 1985.)

One returns closer to the familiar territory of single product systems theory if one considers systems in which there are no 'pure' joint products, in which no process employs more than one type of machine and in which used machines are not transferred between 'industries'. In fixed capital systems of this kind, if free disposal of used machines is assumed then it is immediately ensured that no old machine can have a negative price and it can be proved that 'truncation' – that is, the competitive choice of the ages at which machines are scrapped – will guarantee strictly positive prices for all commodities other than old machines. It can be shown, moreover, that the real wage, measured in terms of any numéraire bundle not containing old machines, will be inversely related to the rate of profit. These results do not depend on the assumption of constant efficiency of machines and, indeed, the theory, by establishing the correct book-value of a machine at each stage of its life, thereby provides an account of the correct depreciation allowances for machines whose efficiency varies over their working lives. In such fixed capital systems, then, the greater complexity of the analysis does not lead to results which are qualitatively different from those obtained in single product systems; the single products results are simply extended. But one cannot, unfortunately, jump to the conclusion that it is only 'pure' joint production which creates complications, fixed capital per se causing no difficulties. For if some processes make joint use of different kinds of machines, or if used machines can be transferred from one 'industry' to another, then in the former case one may have to and in the latter case one will certainly have to make use of a general joint production scheme of analysis: it is only especially restricted forms of fixed capital system that are free from the general complexities of joint production. (*Cf.* Schefold 1976, 1977; the essays by Baldone, Varri and Schefold in Pasinetti (ed.) 1980; Salvadori, 1988.)

The matters touched on thus far in this section could all be seen as issues 'internal' to the development of Sraffa's work. Part I of the present volume concludes, however, with two papers inspired by that work but which discuss some familiar neoclassical concepts and theories. They show how joint products can upset such apparently simple constructions as a wage–rent frontier, the Stolper–Samuelson theorem, the Rybczynski theorem and the effects of Hicksian technical change, even when there are no produced inputs. It is not only Sraffian economics for which joint products cause complications!

3. Other developments

The papers which make up Part II of the present volume cover a wide range of different issues from a broadly Sraffian perspective and will, it is hoped, introduce the reader both to the way in which many areas of economic theory can be illuminated by a Sraffian approach and to a number of further, detailed discussions 'internal' to Sraffian economics. The very breadth of their coverage makes it inappropriate to present here any synoptic view of the arguments and results of all these papers but it may be of help to some readers to comment briefly on three of the topics raised. We may note first the concept of 'vertical integration', which elaborates the concept of a 'sub-system' presented by Sraffa (1960, Appendix A). A vertically integrated sector is a (hypothetical) system which is only part of an actual economic system. The net product of a vertically integrated sector consists of one commodity only, while the inputs (produced and non-produced) to such a sector include all the inputs required, directly and indirectly, to support the production of that net product. Those inputs will typically come from many different sectors of the actual economy but are here (re-)classified together as the inputs to the single (hypothetical) vertically integrated sector. This construction provides a most instructive picture of the direct and indirect conditions of production of the commodity constituting the net product, can therefore be used in the discussion of basics and non-basics (see section 2 above) and shows how the price of each commodity may be 'resolved' into primary incomes plus profits (with no residue). Secondly, we may remind the reader that it is becoming increasingly common to present marginal theory not in terms of production and utility functions but rather in terms of the so-called 'dual' functions – cost functions, profit functions, expenditure functions, etc. There are good reasons for this change in presentation, which is therefore likely to be a durable change and not just a current fashion. Consequently, it is important to demonstrate that the switch to a 'duality' approach does not in any way overcome the Sraffian critique of the marginal theory of value and distribution: two of the papers in Part II seek to contribute to that demonstration. Thirdly, the reader's attention is drawn to the fact that, while no question of returns to scale can ever arise in Parts I and II of *Production of Commodities* (for the simple reason that no change in the absolute level of any input or output quantity is ever considered), the position is not so obviously straightforward in the single Chapter XII of Part III of that work (where changes in technique are considered). And indeed Sraffa's opening paragraph to *Production of Commodities* (p. v) marks the fact, albeit elliptically, that changes in input quantities, at least, may be involved in his Part III discussion. These matters are taken up in the penultimate reading of the present Part II.

Our readings in Sraffian economics conclude with four papers concerned with the history of economic thought. The editorial preface to Sraffa's edition of Ricardo has, in conjunction with *Production of Commodities*, prompted a wide and sometimes sharp debate over various aspects of the history of thought – for example, over the existence or otherwise of a 'corn model' in Ricardo's thought – and that literature alone could by now constitute a substantial set of readings. The papers selected here are simply amongst those which relate most directly to the analysis presented in *Production of Commodities by Means of Commodities.*

Ian Steedman
July 1988

References

Bidard, Ch. (1984), 'Choix Techniques en Production Jointe', in Bidard (ed.) (1984).
Bidard, Ch. (ed.) (1984), *La Production Jointe. Nouveaux Débats*, Paris, Economica.
Jevons, W. S. (1970[1871]), *The Theory of Political Economy*, Harmondsworth, Penguin.
Manara, C. F. (1968), 'Il Modello di Sraffa per la Produzione Congiunta di Merci a Mezzo di Merci', *L'industria*, 1, pp. 3–18. (English translation in Pasinetti (ed.) (1980)).
Pasinetti, L. L. (ed.) (1980), *Essays on the Theory of Joint Production*, London, Macmillan.
Salvadori, N. (1982), 'Existence of Cost-minimizing Systems within the Sraffa Framework', *Zeitschrift für Nationalökonomie*, 42, pp. 281–98. [Chapter 4 in the present volume.]
Salvadori, N. (1984), 'Le Choix de Techniques chez Sraffa: Le Cas de la Production Jointe', in Bidard (ed.) (1984).
Salvadori, N. (1985), 'Switching in Methods of Production and Joint Production', *Manchester School*, 53, pp. 156–78. [Chapter 5 in the present volume.]
Salvadori, N. (1988), 'Fixed Capital within the Sraffa Framework', *Zeitschrift für Nationalökonomie*, 48, forthcoming.
Salvadori, N. and Steedman, I. (1988), 'Joint Production Analysis in a Sraffian Framework', *Bulletin of Economic Research*, 40, pp. 165–95.
Schefold, B. (1976), 'Reduction to Dated Quantities of Labour, Roundabout Processes, and Switches of Techniques in Fixed Capital Systems', *Metroeconomica*, 28, pp. 1–15. [Chapter 6 in the present volume.]
Schefold, B. (1977), 'Fixed Capital as a Joint Product', *Jahrbucher für Nationalökonomie und Statistik*, 192, pp. 415–39.

Schefold, B. (1978), 'On Counting Equations', *Zeitschrift für Nationalökonomie*, 38, pp. 253–85.
Schefold, B. (1980), 'Von Neumann and Sraffa: Mathematical Equivalence and Conceptual Difference', *Economic Journal*, 90, pp.140–56.
Sraffa, P. (1960), *Production of Commodities by Means of Commodities*, Cambridge University Press.
Steedman, I. (1976), 'Positive Profits with Negative Surplus Value: A Reply to Wolfstetter', *Economic Journal*, 86, pp. 873–6.
Steedman, I. (1977), 'Basics, Non-basics and Joint Production', *Economic Journal*, 87, pp. 324–8. [Chapter 8 in the present volume.]
Steedman, I. (1984), 'L'Importance Empirique de la Production Jointe', in Bidard (ed.) (1984).

Part I
Fixed Capital and Joint Production

[1]

The Economic Journal, **92** (*June* 1982), 386–390
Printed in Great Britain

TWO THEOREMS ON JOINT PRODUCTION

Carlo Filippini and Luigi Filippini

The aim of this note is to present some new results in joint production models represented by linear equations.[1] Necessary and sufficient conditions are derived for the existence of a semi-positive (positive) price vector and for a decreasing relation between profit and wage rates.[2] As a corollary the presence of negative labour values and positive prices is explained and the scrap age of a machine determined.

Dominance and positive prices

The assumptions are the usual ones: the system is viable with as many goods as processes; in addition there is one non-produced input – labour – available in unlimited supply. Each process is of unit time duration; longer processes are decomposed introducing if necessary intermediate products as additional goods. Conditions of constant returns to scale prevail and there is equalisation of the rate of profit and of the wage rate. Wages are paid at the end of the production period, but the same results can be obtained assuming pre-payment.

In mathematical terms an economic system using a given technology can be represented by:

$$(1+r)\,\mathbf{pA}+w\mathbf{a}_0 = \mathbf{pB}, \tag{1}$$

or:

$$\mathbf{qC} = \mathbf{0}, \tag{2}$$

where

$$\mathbf{q} = [\mathbf{p} \mid w] \text{ and } \mathbf{C} = \begin{bmatrix} \mathbf{B}-(1+r)\,\mathbf{A} \\ \hdashline -\mathbf{a}_0 \end{bmatrix}.$$

$\mathbf{A}$, $\mathbf{B}$ are the – semi-positive – input and output coefficient $(n \times n)$ matrices, $\mathbf{a}_0$ the – strictly positive – labour input vector, $\mathbf{p}$ the price vector, w the wage and r the profit rate. The scale of the processes can be normalised in many ways: we assume that a technique is said to be activated at the unit level when it uses one worker.

The viability condition is given by:

$$(\mathbf{B}-\mathbf{A})\,\mathbf{y} \geqslant \mathbf{0},$$

for a semi-positive vector $\mathbf{y}$.

It is also assumed that:

$$\det\,[\mathbf{B}-(1+r)\,\mathbf{A}] \neq 0$$

in order to obtain a unique solution for relative prices, given r.

[1] For a general treatment of single product systems see Burmeister and Dobell (1970). For a seminal attempt to deal with joint production systems of linear equations see Schefold (1971); see also his further works on this topic quoted in (1978) and Pasinetti (1980).

[2] We are only concerned with the price system not with the dual, quantity one; even if the economy is viable it may not satisfy a given composition of the demand.

We now specify the idea of dominance. A first meaning – when $r = 0$ – is quite straightforward: if a process yields a greater net output than another one we say that the former dominates the latter. Net output is the $(n+1)$ vector given by the difference between the output and input (including labour) vectors. It is easy to extend this idea to a case where there are more than two techniques: when a non-negative linear combination of processes yields a greater net output than a non-negative linear combination of the remaining ones of a given technology there is dominance.

If $r > 0$ we multiply the inputs by $(1+r)$ (with the exception of labour because we have assumed that wages are paid at the end of the production process); this yields **C**. If the vector of differences between a non-negative linear combination of the columns of **C** and a non-negative linear combination of the remaining columns is semi-positive there is dominance. Prices are not taken into account because we consider each commodity separately.

In other terms there is dominance when:

$$\mathbf{C}_{\mathrm{I}}\mathbf{y}_{\mathrm{I}} \geqslant \mathbf{C}_{\mathrm{II}}\mathbf{y}_{\mathrm{II}}, \tag{3}$$

or: $\mathbf{Cy} \geqslant \mathbf{o}$, where

$$\mathbf{y} = \left(\begin{array}{c} \mathbf{y}_{\mathrm{I}} \\ \hline -\mathbf{y}_{\mathrm{II}} \end{array}\right) \quad \text{and} \quad \mathbf{C} = [\mathbf{C}_{\mathrm{I}} \,|\, \mathbf{C}_{\mathrm{II}}]$$

$$= \left[\begin{array}{c|c} B_i-(1+r)\,A_i & B_j-(1+r)\,A_j \\ \hline -a_{0i} & -a_{0j} \end{array}\right] \quad i = 1 \dots s < n,\ j = (s+1) \dots n.$$

If the inequality holds in a strong sense ($\mathbf{Cy} > \mathbf{o}$) we speak of strict dominance.

Dominance is defined at a given rate of profit so it is possible that in an economic system, defined by a given technology, dominance appears at some values of the profit rate while at others it does not exist.[1] The idea of dominance is related to that of inefficiency. A dominated process yields less than the other(s);[2] in fact if we premultiply $\mathbf{C}_{\mathrm{I}}\mathbf{y}_{\mathrm{I}}$ and $\mathbf{C}_{\mathrm{II}}\mathbf{y}_{\mathrm{II}}$ by a positive vector $[\mathbf{p} \,|\, w]$ we obtain:

$$[\mathbf{p} \,|\, w]\, \mathbf{C}_{\mathrm{I}}\mathbf{y}_{\mathrm{I}} > [\mathbf{p} \,|\, w]\, \mathbf{C}_{\mathrm{II}}\mathbf{y}_{\mathrm{II}}.$$

We now prove the following theorem:

Theorem 1. At a given profit rate there exists a positive (semi-positive) vector of prices if and only if there is no (strict) dominance.

The proof follows from a theorem on linear equations and inequalities: one of the two alternatives holds – either $\mathbf{xF} = \mathbf{o}, \mathbf{x} > \mathbf{o}$ has a solution, or $\mathbf{Fy} \geqslant \mathbf{o}$ does (either $\mathbf{xF} = \mathbf{o}, \mathbf{x} \geqslant \mathbf{o}$ holds, or $\mathbf{Fy} > \mathbf{o}$ holds) (Gale, 1960, pp. 41 ff.). If prices and the wage rate are positive the system (2) has a positive solution and there is no solution to (3), in particular no solution with positive and negative components – that is there is no dominance. To prove sufficiency: *ab absurdo* suppose that at least one price is negative, then (3) has a solution. Three cases are possible: the solution is positive, negative or with mixed components; the first case contradicts the inequality of the system: $-\mathbf{a}_0\mathbf{y} > \mathbf{o}$ (because $\mathbf{a}_0 > \mathbf{o}$);

[1] See Filippini (1977). Wolfstetter (1976) introduces a similar concept in the 2×2 case.
[2] See the R $(= 1+r)$ efficiency assumption in Mirrlees (1969, p. 70).

the second implies that the rate of profit is at least as high as the maximum of the physical expansion rates of the commodities: we have $\mathbf{C}|\mathbf{y}| \leqslant \mathbf{o}$ where $|\mathbf{y}|$ is a vector, the components of which are the absolute values of $\mathbf{y}$; the third case is excluded by hypothesis. This completes the proof. Following similar lines it is possible to prove the other version.

Scrap age for durable goods

The theorem can be used to determine the economic scrap age for durable capital goods, at a given rate of profit. Let us have a simplified model with only a single product and one machine that lasts for several periods. We consider the sub-systems obtained by adding in succession one row and one column to the elements a_{11}, b_{11} (that is considering machines of increasing age). If from one age onwards there is always dominance, that is the scrap age; it is possible to have dominance at a certain age and no dominance with an older machine because its productivity can vary with its age in a complicated way, but when dominance never disappears we scrap the machine.

A numerical example illustrates the point.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 3 & 6 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{a}_0 = [1 \quad 1 \quad 1 \quad 1 \quad 1].$$

Given 3 units of the goods and 1 unit of labour we obtain 5 units of the goods and 1 (new) machine; in successive processes using the machine (at older ages), 2 units of the goods and 1 worker, we get some units of the goods (given by the elements of the first row of $\mathbf{B}$) and the machine (each time one year older).

Let $r = 0$: we can use the same argument for any value of the profit rate, specific results (i.e. the scrap age for the machine) may be different. Matrix $\mathbf{C}$ is now:

$$\mathbf{C} = \left[\begin{array}{ccccc} 2 & 1 & 4 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ \hdashline -1 & -1 & -1 & -1 & -1 \end{array}\right]$$

It is convenient to start the production because one gets a net product of 2 units of the goods (to be distributed as wages). If one then uses the 1-year-old machine the net product of the specific process is only 1 unit: we find dominance because the productivity of the 1-year-old machine is low; but if we go on using the machine productivity rises and dominance disappears; from the third year onwards, however we find dominance.

Negative labour values

Dominance creates a problem for the labour theory of value because in a joint production system one can find positive prices at a positive rate of profit while the labour values of some commodities and the rate of exploitation are negative. In this case the Fundamental Marxian Theorem does not hold (Morishima, 1974). One can define the labour values, $\mathbf{v}$, as the solution of the system:

$$\mathbf{vA}+\mathbf{a}_0 = \mathbf{vB},$$

that is the price system when the rate of profit is zero. The explanation of the problem is simply that at a positive profit rate there may be no dominance, while at a zero rate it may exist.[1]

The wage–profit curve

We turn now to the relation between the profit rate and the wage rate:[2] only a wage–profit curve is considered although the extension to an envelope is trivial. In order to show that the $w-r$ curve is non-increasing it is enough to prove that $d\mathbf{p}^*/dr \geqslant \mathbf{o}$, given $\mathbf{p}^* = \mathbf{p}/w$. In fact:

$$\frac{d(w/\mathbf{p})}{dr} \leqslant \mathbf{o} \quad \text{if and only if} \quad \frac{d(\mathbf{p}/w)}{dr} = \frac{d\mathbf{p}^*}{dr} \geqslant \mathbf{o}.$$

Writing (1) as:

$$\mathbf{p}^*[\mathbf{B}-(1+r)\,\mathbf{A}] = \mathbf{a}_0,$$

we see that:

$$\frac{d\mathbf{p}^*}{dr}[\mathbf{B}-(1+r)\,\mathbf{A}] = \mathbf{p}^*\mathbf{A}.$$

Let us define net yield the vector given by $[\mathbf{B}-(1+r)\,\mathbf{A}]\mathbf{y}$; the scalar product of the net yield by the price vector gives the wage bill.

We now prove the following theorem:

Theorem 2. The $w-r$ curve is non-increasing if and only if the technology cannot be sub-divided into two parts so that in one of these the value of capital goods is lower and the net yield equal or greater than in the other, for $\mathbf{p}^* \geqslant 0$.

The proof is derived from another theorem in linear equation and inequality systems: either $\mathbf{xF} = \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{o}$ holds or $\mathbf{Fy} \geqslant \mathbf{o}$, $\mathbf{by} < 0$ holds (Gale, 1960, pp. 44 ff.). In our case the alternatives are:

$$\frac{d\mathbf{p}^*}{dr}[\mathbf{B}-(1+r)\,\mathbf{A}] = \mathbf{p}^*\mathbf{A}, \tag{4}$$

$$\frac{d\mathbf{p}^*}{dr} \geqslant \mathbf{o},$$

[1] This solves the curiosum raised by I. Steedman. See the seminal article by Steedman (1975), the comment by Wolfstetter (1976) and Steedman's reply to him (1976).

[2] This problem has been considered in two different contexts: on the one hand the treatment is based on the idea of efficiency and optimality in a framework of linear inequalities (Burmeister and Kuga, 1970 and Fujimoto, 1975); on the other hand distribution of the surplus is stressed in a system of linear equations (Filippini and Filippini, 1979).

and

$$[\mathbf{B}-(1+r)\,\mathbf{A}]\,\mathbf{y} \geqq \mathbf{o} \tag{5a}$$

$$\mathbf{p}^*\mathbf{A}\mathbf{y} < \mathbf{o} \tag{5b}$$

where the unknowns are $d\mathbf{p}^*/dr$ and $\mathbf{y}$ respectively. If the system (4) has a solution the technology cannot be sub-divided in the above mentioned way because (5) has no solution. To prove sufficiency we show that (5) cannot have solutions: $\mathbf{y}$ cannot be semi-negative because it means that the rate of profit is at least equal to the highest rate of physical expansion of goods (from (5*a*)); $\mathbf{y}$ cannot be semi-positive because it means that the value of capital goods or inputs is negative (from (5*b*)); $\mathbf{y}$ might be of mixed sign, the system (5) would become:

$$[\mathbf{B}_{\mathrm{I}}-(1+r)\,\mathbf{A}_{\mathrm{I}}]\mathbf{y}_{\mathrm{I}} \geqq [\mathbf{B}_{\mathrm{II}}-(1+r)\,\mathbf{A}_{\mathrm{II}}]\mathbf{y}_{\mathrm{II}}$$

$$\mathbf{p}^*\mathbf{A}_{\mathrm{I}}\mathbf{y}_{\mathrm{I}} < \mathbf{p}^*\mathbf{A}_{\mathrm{II}}\mathbf{y}_{\mathrm{II}}.$$

This means that it is possible to aggregate the processes in two groups so that the net yield of the first group is not less than the second, while input values are lower.

The economic meaning of the condition is that if the $w-r$ curve is increasing we can choose some techniques which can pay at least the same wage bill (not wage rate) as the remaining techniques while using capital goods that have a lower value.

All the above mentioned conditions are similar to those found for efficiency in production. In fact even if the problem is not set in any explicit optimisation framework, some aspects of it are nevertheless present.

Università L. Bocconi, Milano and
Istituto Universitario, Bergamo; Università Cattolica del S.C., Milano

Date of receipt of final typescript: November 1981

REFERENCES

Burmeister, E. and Dobell, A. R. (1970). *Mathematical Theories of Economic Growth*. London: Macmillan.
—— and Kuga, K. (1970). 'The factor–price frontier, duality and joint production.' *Review of Economic Studies*, vol. 37, pp. 11–19.
Filippini, C. (1977). 'Positività dei prezzi e produzione congiunta.' *Giornale degli Economisti e Annali di Economia*, vol. 36, N.S., pp. 91–9.
Filippini, L. and Filippini, C. (1979). 'La relazione tra saggio di salario e saggio di profitto in produzione congiunta.' *Rivista internazionale di Scienze Sociali*, year 87, pp. 425–41.
Fujimoto, T. (1975). 'Duality and the uniqueness of growth equilibrium.' *International Economic Review*, vol. 16, pp. 781–91.
Gale, D. (1960). *The Theory of Linear Economic Models*. New York: McGraw-Hill.
Mirrlees, J. A. (1969). 'The dynamic nonsubstitution theorem.' *Review of Economic Studies*, vol. 36, pp. 67–76.
Morishima, M. (1974). 'Marx in the light of modern economic theory.' *Econometrica*, vol. 42, pp. 611–32.
Pasinetti, L. L. (ed.) (1980). *Essays on the Theory of Joint Production*. London: Macmillan.
Schefold, B. (1971). 'Mr Sraffa on joint production.' Basel. (Private print.)
—— (1978). 'On counting equations.' *Zeitschrift für Nationalökonomie*, vol. 38, pp. 253–85.
Steedman, I. (1975). 'Positive profits with negative surplus value.' ECONOMIC JOURNAL, vol. 85, pp. 114–23.
—— (1976). 'Positive profits with negative surplus value: a reply to Wolfstetter.' ECONOMIC JOURNAL, vol. 86, pp. 873–6.
Wolfstetter, E. (1976). 'Positive profits with negative surplus value: a comment.' ECONOMIC JOURNAL, vol. 86, pp. 864–72.

[2]

SOME PROBLEMS CONCERNING THE NOTION OF COST-MINIMIZING SYSTEMS IN THE FRAMEWORK OF JOINT PRODUCTION*

by
REINER FRANKE†
University of Bremen

Introduction

The present paper is concerned with the issue of production prices and the underlying notion of equilibrium in linear joint production systems when the non-substitution theorem no longer applies. One could, of course, consider a given quadratic system of production processes and derive necessary conditions on the structure of demand such that it can be satisfied by an appropriate combination of these processes. Or, in other words, one could just postulate it to be contained in the net output cone. But this approach, on the one hand, leaves aside the question why it is just this system that has come into existence and, on the other hand, leaves unresolved how it is that demand is determined in such a way as to comply with the desired properties. To treat this issue in detail, the problem of the choice of techniques has to be discussed explicitly and the demand side has to be brought in right from the beginning. This leads to a certain notion of equilibrium which, following Salvadori (1982), we shall call a cost-minimizing system.

As a basis, input and output matrices A and $B \in \mathbb{R}_+^{n \times m}$ represent the production of n products in m processes ($m \neq n$ is admitted). Labour is supposed to be homogeneous and $l \in \mathbb{R}_+^m$ is the corresponding vector of direct labour inputs. Activity levels are denoted by column vectors $x \in \mathbb{R}_+^m$, prices by row vectors $p \in \mathbb{R}^n$ (prices are not necessarily non-negative). As for distribution, a rate of profit $r \geq 0$ is considered as given. Wages are paid *post*

*Manuscript received 4.9.84; final version received 4.4.85.

†For helpful remarks I wish to thank U. Krause, N. Salvadori and I. Steedman, as well as an anonymous referee.

factum. We restrict our interest to situations in which they are positive and thus the nominal wage rate can be set equal to unity, i.e., all prices are in terms of labour commanded. The demand side, finally, will be represented by a function $d = d(p, x)$, $d \in \mathbb{R}^n$, of "requirements for use", to employ Sraffa's wording (*cf*. Sraffa, 1960, p. 43n). It is to be thought of as net of reproduction.

With respect to these data a *cost-minimizing system* is defined as a position in which all operated production processes earn the given rate of profit r, no non-operated process would yield surplus profits, and the resulting demand can be fulfilled. Formally, it is a pair $(p, x) \in \mathbb{R}^n \times \mathbb{R}^m_+$ such that

$$pB \leqq (1+r)\, pA + l \qquad \ldots\ldots(1)$$

$$pBx = (1+r)\, pAx + lx \qquad \ldots\ldots(2)$$

$$Bx = Ax + d(p, x) \qquad \ldots\ldots(3)$$

(This definition is a bit different from that of Salvadori, 1982 who, in addition, takes into account compatible price systems.)

In order for this notion to be meaningful, the first question at issue relates to existence. As the definition appears quite natural one might expect that the conditions on technology and demand by which existence is guaranteed are not unreasonable. The first section of this paper is devoted to a discussion of this issue. Whereas the approach introduced there seeks a high level of generality, the subsequent sections deal with more specific questions. Section II is concerned with an interpretation of negative prices and the role of costly disposal. Section III discusses a special phenomenon, namely the possibility of a cost-minimizing system without capital, in the usual sense. In the final section the notion of a cost-minimizing system is applied to the issue of a wage-profit frontier. We give a simple example of an economy generating a frontier that exhibits a peculiarity of a new type.

I

It is trivial that even if the technology (B, A) can produce a positive net output, for some function $d(.\,,\,.)$ a cost-minimizing system will not exist: just suppose that the net output cone does not contain the whole positive orthant and that $d(p, x)$ is always to be found in the complementary region. But there are other counter-examples which are less obvious (*cf*. Salvadori, 1982, p. 287). The unpleasant phenomenon of non-existence may also occur for the following most elementary function

$$d(p, x) = g\, Ax + c \qquad \ldots\ldots(4)$$

where g is a given growth rate, $0 \leqslant g \leqslant r$, and the consumption vector $c \in \mathbb{R}^n_+$ can in fact be produced by some combination of processes of (B, A). This is due to the possibility that if c is contained in the g-net output cone of a, say quadratic, technique (by which we mean a set of production processes), then (with $g \neq r$), the corresponding vector of production prices cannot prevent some processes outside this technique from being more profitable. And the other way round, any technique supported by a price vector that rules out surplus profits is incapable of producing c as a g-net output. A concrete example of this was provided in Salvadori (1985).

So the problem is to find conditions on (B, A, l) and $d(.\,,.)$ which are able to bring into accord these two different sides. Since this question is of central importance we shall devote some space to it (and in order to be self-contained we take the risk of some repetition from Franke, 1984).

Salvadori himself has pointed out two possibilities. According to him a cost-minimizing system does exist

1. with respect to the function (4) if free disposal is admitted for all commodities, i.e., if all prices are non-negative and if (3) is relaxed to admit "$\geqslant$", where, in exchange, $pBx = pAx + pd(p, x)$ is required (see Salvadori, 1980, Theorem 4.2., p. 59);
2. or if there exists a commodity $u \in \mathbb{R}^n$ that has certain properties with respect to $d(.\,,.)$ and certain sets of prices and activity levels (here $d(.\,,.)$ need not be restricted to functions of the type (4)). Important cases for which existence can be proved in this way are $g=r$, on the one hand, and a sufficiently amenable kind of joint production like r-all-productive, r-all-engaging or r-s-partially-all-productive systems, on the other (see Salvadori, 1982).

Aiming at a higher level of generality, in Franke (1984) a different approach for solving the existence problem was set out. Its most significant feature is the assumption that the function $d(.\,,.)$, over its whole domain, obeys "Walras's Law", i.e., the identity

$$pBx = pAx + pd(p, x);$$

if, with respect to a time-discrete period model, p are the prices of the present production period and Bx is the vector of last period's gross output, this gives rise to a gross income pBx, out of which this period's demand $Ax + d(p, x)$ is to be defrayed.

A second point to be made is that for some selected commodities free disposal may be admitted (if their price turns out to be zero) or that, in another more stringent version, we wish a positive price to be associated with them. These commodities may be composite ones and they may even vary with the activity vector x. As for technology, it is postulated that the whole positive orthant is contained in the r-net output cone of (B, A) – which, by the way, holds true for single production systems whose maximal rate of profit exceeds r. In addition, all processes are costly in the sense that they require a positive amount of labour.

In detail, the assumptions are as follows. In a preparatory step, fix a given level of employment $L > 0$ and define

$$X := \{x \in \mathbb{R}^m_+ : lx = L\}.$$

Furthermore, let J be a set of indices and, for each $j \in J$, $c^j(.)$ be a function from X to $\mathbb{R}^n_+$.

Assumption 1

The set J is non-void and finite, $1 \in J$, and $c^1(x) = c^1 =$ constant. All functions $c^j(.): X \longrightarrow \mathbb{R}^n_+$ are semi-positive and continuous.

The assumption is, in the first place, adopted for mathematical reasons of proof (it allows us to utilize an extended version of the Gale-Nikaido-Debreu Lemma; this hint also explains that all functions involved are assumed to be continuous). On the other hand, it will exclude the strange event that in a cost-minimizing system all prices happen to be negative, which, in the light of the positive wage rate $w = 1$, would lead to difficulties in interpretation (*cf.* Section II below).

Assumption 2

(i) $l_k > 0, \qquad k = 1, \ldots, m.$

(ii) For all $c \in \mathbb{R}^n_+$ there exists an $x \in \mathbb{R}^m_+$ such that $(B - (1 + r)A)x = c$.

(iii) There exists a neighbourhood V of c^1 such that for all $c' \in V$ an $x' \in \mathbb{R}^m_+$ can be found with $(B - (1 + r)A)x' = c'$.

Assumption 2(iii) is a bit stronger than 2(ii) in that it requires us to allow a negative component i in the vector $c' \in V$ if $c^1_i = 0$. In other words, c^1 is supposed to be contained in the interior of the r-net output cone, whereas the other commodities $c^j(x)$ need not be.

As for the assumptions on $d(. , .)$, two cases are distinguished, according as to whether free disposal is admitted for all commodities $c^j(x)$ in J or whether we want it to be excluded for a subset J' of J. Define the sets

$$D := \{(p, x) \in \mathbb{R}^n \times X : (p, x) \text{ satisfies (1) and (2)}, pc^j(x) \geqslant 0 \text{ for all } j \in J\},$$

and, with respect to the subset J',

$$D' := \{(p, x) \in D : pc^j(x) > 0 \text{ for all } j \in J'\}.$$

Assumption 3.1

$d(. , .)$ is a continuous function $D \longrightarrow \mathbb{R}^n$ for which $pBx = pAx + pd(p, x)$ holds identically on D.

In contrast, Assumption 3.2 postulates that if for $j \in J'$ the price of a commodity $c^j(x)$ is sufficiently low, then so much (of this commodity or of another) is demanded that it cannot be produced with the given level of employment L. More specifically:

Assumption 3.2

There is a non-void subset $J' \subset J$ such that with respect to the correspondingly defined D' the following holds:

(i) $d(.\,,\,.)$ is a continuous function $D' \longrightarrow \mathbb{R}^n$ which is bounded from below;

(ii) $pBx = pAx + pd(p, x)$ for all $(p, x) \in D'$;

(iii) there exists an $\epsilon > 0$ with the property: $(p, x) \in D'$ and $pc^j(x) < \epsilon$ for some $j \in J'$ implies the existence of some i such that $(Bx)_i + \epsilon < (Ax)_i + d_i(p, x)$.

A function of the rigid type (4) could be incorporated in this framework by defining

$$d(p, x) = g\,Ax + \left[\frac{p(B - (1 + g)A)x}{pc}\right]c \qquad \text{......(4')}$$

Assumption 3.2 is fulfilled if c is present in Assumption 1 and in the definition of D', say $c^1 = c$ and $1 \in J'$, and if on D' $p(B - (1+g)A)x$ is positively bounded away from zero. Because of $p(B - (1+g)A)x = p(B - (1+r)A)x + (r-g)$ $pAx = lx + (r-g)\,pAx$, this will hold true in the following cases:

$g = r$ or g sufficiently close to r;

$c^j(x) = Ax$ for some $j \in J$;

all commodities can be freely disposed of, i.e., in formal terms, if $J = \{1, \ldots, n+1\}$, all functions c^j are constant, $c^1 = c$, $c^{j+1} = j$-th unit vector for $j = 1, \ldots, n$, $J' = \{1\}$, so that the domain of $d(.\,,\,.)$ becomes

$$D' = \{(p, x) \in \mathbb{R}^n \times X\colon (p, x) \text{ satisfies (1) and (2)}, p \geqslant 0, pc > 0\}.$$

Theorem

(i) Suppose that Assumptions 1, 2, and 3.1 hold. Then there exists a pair $(p, x) \in D$ such that

$$Bx = Ax + d(p, x) + c^* \qquad \text{......(5)}$$

where $c^* = \sum_{j \in J} \lambda_j\, c^j(x)$ for some $\lambda_j \geqslant 0$, and $\lambda_j > 0$ implies $pc^j(x) = 0$.

(ii) Suppose that Assumptions 1, 2, and 3.2 hold. Then there exists a pair $(p, x) \in D'$ bringing about (5) and $\lambda_j = 0$ for all $j \in J'$.

In particular, if $J' = J$ and Assumption 3.2 applies, then all λ_j vanish and, in fact, (3) is satisfied.

In summary, the approach put forward could be termed one of general equilibrium. Though general equilibrium theory is often considered to be the hallmark of neoclassical economics, this does not necessarily signify that our model and its assumptions are inconsistent with a classical mode of thought (as for the "visions" underlying classical and neoclassical modelling we confine ourselves to referencing A. J. and J. S. Cohen, 1983, pp. 194-200).

In particular, consumption demand need not be the only relevant component of the function $d(.\,,.)$. Moreover, since prices as well as activity levels enter as arguments, it provides a black box in which a variety of demand concepts could find their place (and they may be rather distinct from that of utility maximizing individuals).

II

The definition of a cost-minimizing system does not require all prices to be non-negative. Apart from the commodities $c^j(x)$, the existence theorem does not say anything more specific about this, either. However, some economists seem to think along the lines that, in the long run, only production processes which bring about non-negative prices will survive (or else the uniformity of the wage rate or the rate of profit can no longer be maintained). There need not necessarily be a contradiction here (neither has the postulate of uniformity to be given up). We shall argue that it suffices to modify the equations for the prices of production (in the activated processes) appropriately.

The starting point is to make disposal activities explicit: to get rid of a thing one has to make use of a production process and put it in there, as an input. This may be a special process that produces no output at all, but any other "normal" process will do (whether it is tolerated by the law need not matter). Thus, a negative price p_i makes sense if $-p_i$ is interpreted as the payment for a service, namely the service of disposing of one unit of product i in a production process k which entails some costs, in particular for the labour input $l_k > 0$. A corresponding redefinition of negatively priced products would make all negative prices disappear from the economy.

In writing down the production price equations, however, care has to be taken of the dating: the waste is delivered at the beginning of the period and it is already at this point of time that the service of its subsequent disposal has to be paid for. So, let I^+ and I^- denote the sets of indices of positive and negative prices, respectively. Then the production price equation of the k-th process can be rearranged to read

$$(1+r)\ \textstyle\sum_{i\epsilon I^+} p_i a_{ik} + \sum_{i\epsilon I^-} |p_i| b_{ik} + l_k = \sum_{i\epsilon I^+} p_i b_{ik} + (1+r) \sum_{i\epsilon I^-} |p_i| a_{ik},$$

with costs, with respect to the end of the period, on the left and revenues on the right.

Note that whether the price of a certain product i turns out to be positive or negative in equilibrium, i.e., whether it turns out to be a good or a bad, is not only a matter of the structure of demand, but also depends upon distribution. A variation of the rate of profit will change p_i and it is perfectly possible that this has an impact on its sign as well.

If one follows this rationale for negative prices, then the idea that a certain commodity or service c is a consumption good will surely not be consistent with a situation $pc < 0$. This has been an economic reason for introducing Assumption 1 and the definition of the sets D and D'. A stronger version of the idea of an *a priori* consumption good can be grasped by Assumption 3.2(iii).

As regards the problem of disposal, even if free disposal is considered to be a hardly defensible supposition (and there are good reasons for this opinion), it may be acceptable to permit it for some few selected products, as they are specified in the goods $c^j(x)$ of Assumption 1. Part (ii) of the Theorem shows that by employing Assumption 3.2(iii) for all commodities $c^j(x)$, free disposal can be made completely absent in a cost-minimizing system—a sufficient reason being that there is no need for it. It might seem, at first sight, that this assumption also excludes any need for disposal in general. Observe, however, that besides the "normal" components of demand, demand for consumption and demand for means of production, the function $d(.\,,.)$ may equally include a "demand for bad i", to which, more meaningfully, corresponds the supply of the service of disposing of product i. Accordingly, on the left-hand side of (3) we may have the supply of a bad i, to which corresponds the demand for this service: for the sake of some other advantages, product i has been produced in the preceding period (as a by-product of a positively priced commodity), and in an equilibrium position now, at the beginning of the present period, by definition there has to be someone who is willing to take it. However, the actual disposal is subsequent and no longer an explicit part of the model.

III

The main idea behind the definition of a cost-minimizing system is, of course, to model some basic features of a capitalistic economy. Now, if $(pA)_k \leqslant 0$ for some production process k in such a system, the question arises as to what has become of the notion of capital in that process. Two points of view seem possible. Suppose

$K_1 := \sum_{i \in I^+} p_i a_{ik} > 0$, $K_2 := \sum_{i \in I^-} |p_i| a_{ik} > 0$, $K_1 - K_2 < 0$, I^+ and I^- still being the sets of indices of positive and negative prices, respectively. According to the first point of view, an operator of process k, on the whole, receives a positive sum of money $K_2 - K_1$, at the beginning of the period and invests it in some other processes that earn a rate of profit r. At the end of the period he thus has a (gross) revenue $(pB)_k + (1+r)(K_2 - K_1)$, and this is just sufficient to cover the wage bill, l_k.

On the other hand, he himself may advance the sum K_1, investing elsewhere all the payments K_2 received for his disposal services. Then his net revenue amounts to $(pB)_k + (1+r)K_2 - l_k = (1+r)K_1$. Here K_1 could be regarded as capital, whereas in the first case it has to be looked for elsewhere. However, the point to be made is that in both cases a positive profit has to be realized outside k. This might not be too serious, but what of the notion of (total) capital and profits if, economy-wide, $pAx \leqslant 0$ happens to come about?

In fact, in the absence of more detailed information it cannot be excluded that a system fulfilling equations (1)-(3) is incompatible with the classical view of a capitalistic economy. Just consider the extreme case that $(pA)_k = 0$ for all operated processes, and $(pA)_k > 0$ for all processes that are not operated (because of $(pB)_k < (1+r)\ (pA)_k + l_k$). Since in the classical tradition the ultimate reason for a positive rate of interest is to be found in the sphere of production, namely in the need of some production processes to finance their positively priced inputs, r cannot be interpreted as a rate of interest or as being in a certain relationship to it. By the same token, it makes no sense to speak of profits and a rate of profit. On the contrary, apparently the sole task of the number r is to prevent activation of the processes with $(pA)_k > 0$.

We may conclude that, as a minimum, it is desirable that $pAx > 0$ in a cost-minimizing system. This could be guaranteed by imposing Assumption 3.2(iii) not only on some consumption goods, but also on $c^j(x) = Ax$. The justification for the former is that, at prices rendering pc sufficiently small, the good c itself is demanded in such an amount that it cannot be produced with the given level of employment. As for $c^j(x) = Ax$, the argument turns on the structure of demand, rather than on scale. It entails that for small pAx there is always one component i such that the demand $(Ax)_i + d_i(p, x)$ exceeds the existing quantity $(Bx)_i$ of this product. Which one does not matter. It is quite possible that this i changes as p and x vary in the set, say, $\{(p, x) \in D: pAx = \epsilon\}$ for some small $\epsilon > 0$. This assumption, however, requires a certain flexibility of the function $d(.\,,.)$, especially if the negatively priced products are not pure consumption goods.

IV

The existence theorem of Section I does not make any statement on the uniqueness of a cost-minimizing system, not even in a local sense. Likewise, one does not learn anything about the number of activated processes, e.g., whether it is equal to the number of commodities actually produced and traded, in which case it may be called a "Sraffian" system. Now, examples of multiple or non-quadratic equilibria can be readily constructed. But such an event is more than just a matter of pure accident or of aesthetics, as will be seen when applying the notion of a cost-minimizing system to the issue of a wage-profit frontier.

If a wage-frontier is considered at all in the framework of joint production *and* choice of techniques, it is often limited to a special case of requirements for use, namely the function $d(.\,,.)$ of (4) where free disposal and $g=r$ are both assumed. On the other side, the analysis uses the instrument of linear programming, and this already enters into the very definition of the frontier. If $g < r$ is admitted, however, it seems more natural to start from the notion of a cost-minimizing system and assign to each rate of profit r the real wage rates of all systems existing with respect to that r. The wage-profit frontier can, then, be defined as the locus of the maximal wage rates.

Let us apply this procedure to the following example of two commodities and two production processes,

$$B = \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, l = (1, 1), c = \begin{bmatrix} 10 \\ 11 \end{bmatrix}, g = 0.$$

A cost-minimizing system exists for r contained in the interval $[0, 4)$. Fixing $L=7$, on the interval $[0, 3)$ one system is given by $x_1(r) = 5, x_2(r) = 2$,

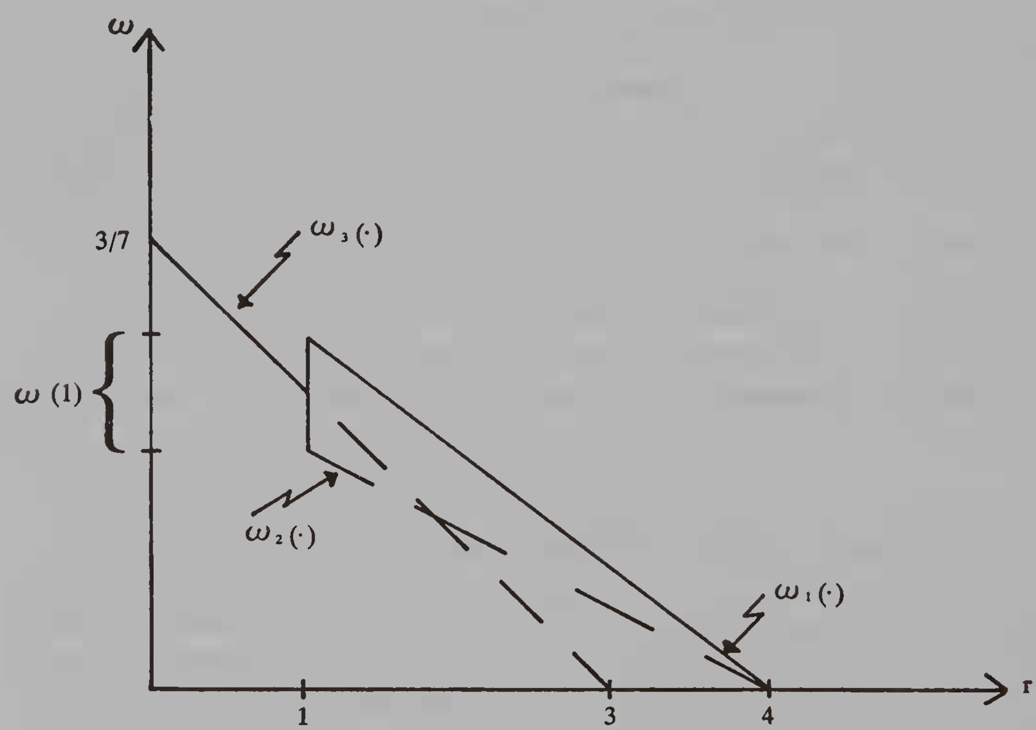


Fig. 1.

and $p_1(r) = p_2(r) = 1/3(3-r)$. On [0, 1) this is the only one. At $r=1$, a continuum of prices corresponds to it, namely the set $\{(p_1, p_2) \geqslant 0: p_1 + p_2 = 1/3\}$. Maintaining the assumption of free disposal, for $1 \leqslant r < 4$ two additional systems appear, one in which exclusively process 1 is operated, with $p_1(r) = 1/(4-r)$, $p_2(r) = 0$; and, symmetrically, another in which only process 2 is operated, with $p_1(r) = 0$, $p_2(r) = 1/(4-r)$.

The real wage rate is given by $\omega = \omega(r) = 1/p(r)c$. Define $\omega_3(r) := (3-r)/7$, $\omega_1(r) := (4-r)/10$, $\omega_2(r) := (4-r)/11$, i.e., the real wage rate at r if both processes, process 1, and process 2, respectively, are activated. All possible wage rates are registered by the set-valued mapping $\omega = \omega(r)$ defined by

$$\omega(r) := \begin{cases} \omega_3(r) & \text{for } 0 \leqslant r < 1 \\ [\omega_2(1), \omega_1(1)] & \text{for } r = 1 \\ \{\omega_1(r), \omega_2(r), \omega_3(r)\} & 1 < r < 3 \\ \{\omega_1(r), \omega_2(r)\} & 3 \leqslant r < 4 \end{cases}$$

which is illustrated in Fig. 1. In particular,

$$\omega_2(1) < \omega_3(1) < \omega_1(1).$$

To sum up, each separate wage curve $\omega_k(.)$, $k = 1, 2, 3$, is a falling real function of r. However, at $r=1$ possibly a discontinuous change of prices occurs to which there may or may not correspond a switch of techniques. As a consequence, a slight increase in the rate of profit can induce a sudden rise in the real wage rate.

References

Cohen, A. J. and Cohen, J. S. (1983). "Classical and Neoclassical Theories of General Equilibrium", *Australian Economic Papers*, Vol. 22, No. 40, pp. 180-200.

Franke, R. (1984). "Joint Production and the Existence of Cost-Minimizing Systems", University of Bremen, mimeo.

Salvadori, N. (1980). "On a Generalized von Neumann Model", *Metroeconomica*, Vol. 32, No. 1, pp. 51-62.

——— (1982). "Existence of Cost-Minimizing Systems within the Sraffa Framework", *Zeitschrift für Nationalökonomie*, Vol. 42, No. 3, pp. 281-298.

——— (1985). "Switching in Methods of Production and Joint Production", *The Manchester School*, Vol. 53, No. 2, pp. 156-178.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge, Cambridge University Press.

[3]

DEPRECIATION OF MACHINES OF CHANGING EFFICIENCY: A NOTE*

GEOFF HODGSON

Manchester Polytechnic

IAN STEEDMAN

University of Manchester

In analyses of the depreciation of fixed capital, it is common'y assumed that the efficiency of the plant or equipment in question is independent of its age. In this note, the effects of relaxing that assumption are examined in the context of two different types of depreciation analysis, the first being that adopted by Marx [4], [5] and the second that associated with Champernowne and Kahn [1], Sraffa [8] and others.[1] Our general approach will be the one employed by von Neumann [6] and Sraffa [8], Part II, in which machines of different ages are regarded as distinct commodities and the use of fixed capital is then treated as a special case of joint production, an "old machine" being produced jointly with the "product" in the normal sense.

The Methods of Production

We shall consider an extremely simple fixed capital model, in which the only produced means of production is a machine which lasts for two periods of production (two "years"). The new and old machines are used, by homogeneous labour, to produce the single consumption commodity, corn, but the new machine is itself produced by unassisted labour. Constant returns to scale prevail in each of the production processes, the input and output coefficients for which are shown in Table I, with input coefficients to the left of the arrows and output coefficients to the right.[2] The third row of the Table shows that, by

TABLE I

New Machines	*Old Machines*	*Labour*		*New Machines*	*Old Machines*	*Corn*
–	–	*a*	→	1	–	–
b	–	*c*	→	–	*b*	1
–	1	1	→	–	–	1

*We would like to thank G. C. Harcourt for helpful comments.

[1] While Marx made some illustrative references to changing efficiency of machines, he does not appear to have let such considerations affect his treatment of depreciation. Champernowne and Kahn explicitly assume constant efficiency throughout, as does Robinson [7]. Hudson and Mathews [3] consider the case of varying efficiency but do not note that, as will be shown below, depreciation can be negative, while the complete "capital charge" may be better-behaved than the depreciation element alone. Sraffa presents a completely general method for treating fixed capital, which permits any pattern of changing efficiency, but his discussion of several points is confined to the constant efficiency case (see [8], pp. 68–72).

[2] While the simplest procedure is to assume constant returns to scale, it is not strictly necessary to do this; we consider no changes in output levels and thus the coefficients in Table I may just be regarded as the *average* coefficients obtaining with the given output levels.

an appropriate choice of units, one unit of labour works with one *old* machine to produce one unit of corn. The second row shows that c units of labour work with b *new* machines to produce one unit of corn and also, as an inevitable by-product, b old machines. As is shown in the first row, the number of units of unassisted labour required for the production of each new machine is a.

It will be clear that if $b = c = 1$ then the machine is of constant efficiency throughout its life. If, however, $b \geqslant 1$ and $c > 1$ or $b > 1$ and $c \geqslant 1$ then the machine is less efficient when new than when one year old. While if $b \leqslant 1$ and $c < 1$ or $b < 1$ and $c \leqslant 1$ the efficiency of the machine is falling with age. (We shall not attempt to say how efficiency is changing for the remaining cases $b < 1$, $c > 1$ and $b > 1$, $c < 1$.) It is important to note that in the case of rising efficiency, due perhaps to the benefits of "running-in", we shall assume that the increase in efficiency results from the *use* of the new machine and not simply from its ageing.[3]

Prices and Profits

Let r be the annual rate of profit and p_n, p_0 and p_c be the corresponding labour-commanded prices of new machines, old machines[4] and corn respectively. Since a labour-commanded price is, of course, the price of a commodity relative to the wage rate, it follows that the reciprocal of p_c is the real wage rate in terms of corn. If wages are paid at the end of each year, rather than in advance, then in equilibrium the following relations must hold (*c.f.* Table I),

$$a = p_n , \tag{1}$$

$$(1+r)\, bp_n + c = bp_0 + p_c , \tag{2}$$

$$(1+r)\, p_0 + 1 = p_c . \tag{3}$$

Defining $R \equiv (1 + r)$, it follows from (1), (2) and (3) that

$$p_n = a , \tag{4}$$

$$p_0 = \left[\frac{(c-1) + ab\,R}{b + R}\right] \tag{5}$$

$$p_c = \left[\frac{b + cR + ab\,R^2}{b + R}\right] \tag{6}$$

(If wages are paid at the beginning of the year, rather than at the end, then (4), (5), (6) still hold provided that, in each case, the right-hand side is multiplied by R.)

It will be seen from (4) and (6) that p_n and p_c are necessarily positive but the situation is more complex with respect to p_0, the price of a used machine. Since the new machine is

[3] If this assumption were not made then a possible technique for the production of corn would be that of producing a new machine, leaving it to "mature" for one year and then using it to produce corn for one year; we should then have to examine whether or not it would be profitable to adopt this technique. (Such a technique is, of course, available even when efficiency is not rising but it is obvious that in such cases this technique would not be used.)

[4] If there exists a fully developed market in one year old machines, with zero transaction and transport costs, then p_0 is an equilibrium market price; whether or not such a market exists, p_0 is the correct book value to be attached to an old machine by the capitalist using it.

produced by unassisted labour, there is no finite maximum profit rate in this model. Now as r increases without limit, it is clear from (5) that p_0 approaches ever closer to ab, so that p_0 is always positive for sufficiently high rates of profit. For low rates of profit, however, p_0 can be negative, according to (5), if $c < (1 - ab)$; we shall assume that $c > (1 - ab)$, since we do not wish to discuss, in this note, the complications that arise when this assumption is not made.[5]

Marx's Analysis of Value Depreciation

In his analysis of the value of a commodity, Marx did not normally adopt the procedure, used above, of both entering the machine as input to the production process and entering a one-year-older machine as an output from it. Instead he usually adopted the convention of merely entering the depreciation of the machine on the input side, apparently considering this procedure equivalent to, though simpler than the other. Thus he wrote in *Capital,* vol. I;

"The two things compared are, the value of the product and the value of its constituents consumed in the process of production. Now we have seen how that portion of the constant capital[6] which consists of the instruments of labour, transfers to the product only a fraction of its value, while the remainder of that value continues to reside in those instruments. Since this remainder plays no part in the formation of value, we may at present leave it on one side. To introduce it into the calculation would make no difference . . . Throughout this Book therefore, by constant capital advanced for the production of value, we always mean, unless the context is repugnant thereto, the value of the means of production actually consumed in the process, and that value alone". ([4], pp. 212–213.)

The value of a commodity, as defined by Marx, is the total labour required, directly and indirectly, for the production of that commodity.[7] It is clearly envisaged in the above quotation that the older machine will be of smaller value than the newer (hence the references to the *fraction* and to the *remainder* of the initial value) and certainly in all of his many numerical examples Marx assumed the depreciation in value to be positive. This indeed followed from the fact that he assumed *straight-line* depreciation.[8] We shall see below, however, that straight-line depreciation is not, in general, appropriate in Marx's value depreciation analysis and that value depreciation can actually be negative.

[5] It can be shown that if the Marxian value of an old machine, ℓ_0, (and hence p_0) are positive then the third process, the old machine using process, will always be used – the machine will not be scrapped after one year. This, together with the assumption that "maturation" is not possible (see fn.3 above), ensures that all three processes will be operated at positive levels. $\ell_0 > 0$ is also a sufficient, though not necessary, condition for $(dp_c/dr) > 0$, *i.e.*, for the real corn wage to be inversely related to r.

[6] Marx uses the term "constant capital" here to refer to inputs other than labour; it includes both fixed and circulating capital elements, the former being "the instruments of labour"

[7] More precisely, it is the "socially necessary, abstract labour" required.

[8] See, for example, *Capital*, vol. II [5], p. 161. "But whatever may be [an instrument's] durability, the proportion in which it yields value is always inverse to the entire time it functions. If of two machines of equal value one wears out in five years and the other in ten, then the first yields twice as much value in the same time as the second."

The value of a commodity, in Marx's sense, is equal to the labour commanded by that commodity when profits are zero, so that on defining ℓ_n, ℓ_0 and ℓ_c as the values of a new machine, an old machine and a unit of corn respectively and setting $R = 1$ in (4), (5) and (6), we find that

$$\ell_n = a \tag{7}$$

$$\ell_0 = \left[\frac{ab + c - 1}{b + 1}\right], \tag{8}$$

$$\ell_c = \left[\frac{(1 + a)\, b + c}{b + 1}\right]. \tag{9}$$

It will be clear from (8) that in the constant efficiency case, ($b = c = 1$), $\ell_0 = a/2 = \ell_n/2$, so that straight-line depreciation is correct. More generally, it follows from (7) and (8) that $\ell_0 = \ell_n/2$ if and only if $a\,(1 - b) = 2(c - 1)$; clearly there is no reason why this con dition should hold and indeed it *cannot* hold for either rising or falling efficiency. This point may also be seen as follows:

It is obvious from the first row of Table I that $\ell_n = a$. Now using a linear depreciation approach, we can obtain ℓ_c from either the second or the third row of the table. Thus

(from second row), $b\,(a/2) + c = \ell_c$

(from third row), $(a/2) + 1 = \ell_c$.

These two results for ℓ_c are inconsistent, however, unless $a(1 - b) = 2(c - 1)$.

In the production process using the new machine, the correct depreciation in Marx's value would be $(\ell_n - \ell_0)$ per machine, which, from (7) and (8), is given by

$$(\ell_n - \ell_0) = \left[\frac{(1 + a) - c}{b + 1}\right] \tag{10}$$

Now it is clear from (10) that if $c > (1 + a)$ then $\ell_n < \ell_0$; the old machine has a *greater* amount of labour embodied in it than has the new one and hence Marx's value depreciation for the new machine using process would have to be *negative*. Before attempting to give some explanation of these results, we may note that the condition $c > (1 + a)$ is incompatible with either constant or decreasing efficiency.

The Meaning of $\ell_0 > \ell_n$.

If the result $\ell_0 > \ell_n$ seems strange then it may help to note first that the only way to "produce" an old machine is to operate the process shown in the second row of Table I. Since this involves employing *(c/b)* units of labour per new machine, in addition to the "*a*" units used in making the new machine, it becomes plausible that ℓ_0 might exceed ℓ_n. The real difficulty lies, however, in thinking of the new and old machine as "essentially" the same product, the difference in age being a secondary consideration. But age is not a trivial distinction between the two machines, even when efficiency is constant; *a fortiori* it is not a trivial one when efficiency is not constant. Once we accept that new and old machines are truly different commodities it becomes clear that there is no more reason to be surprised by $\ell_0 > \ell_n$ than by $\ell_n > \ell_0$ or $\ell_n > \ell_c$ or $\ell_c > \ell_n$. From this standpoint the flaw in Marx's treatment of depreciation is not so much that he assumed $(\ell_n - \ell_0)$ to be positive as the fact that he tried to deal with depreciation in *net* terms at all. If the values of commodities are to be analysed in terms of Marx's value accounts, then new and old machines must be treated as distinct commodities, each with its own value, and must not be treated as a single commodity that gradually yields up its value in the form of depreciation.

Prices of New and Old Machines

We turn now to the more usual interpretation of depreciation, *i.e.*, not the difference in Marx's value as between new and old machines, but rather the difference between the prices of new and old machines. It is well-known that for a constant efficiency machine with a total life of two years, the price of the old machine relative to that of the new one will be exactly one half when profits are zero and will rise monotonically as r increases from zero. When efficiency is not constant, however, no such definite *a priori* results are available.

As for the relative price of old and new machines at zero profits, it has already been shown above, equation (10), that ℓ_0 can even exceed ℓ_n so that $p_0 > p_n$ for $r = 0$. More generally, at $r = 0$, $p_0 \gtreqless \frac{1}{2} p_n$ is possible.

More interesting, perhaps, is the fact that, with changing efficiency, the relative price of an old machine need not increase with r. To see this. since $p_n = a$ is independent of r, we need only differentiate (5) to find that

$$\text{Sign } \frac{dp_0}{dr} = \text{Sign } [(1 + ab^2) - c] \; . \tag{11}$$

If efficiency is constant or decreasing then (11) indeed implies that $\frac{dp_0}{dr} > 0$ but in some cases p_0 can fall, or even remain constant, as r rises. In the particular case $c = (1 + ab^2)$, for example, it is easy to show that

$$p_n = a \; , \tag{4'}$$

$$p_0 = ab \; , \tag{5'}$$

$$p_c = (1 + abR) \; ; \tag{6'}$$

the old machine has a constant price relative to the new machine and even, if $b > 1$ in addition, a greater absolute price for all rates of profit. Leaving aside this special case $c = (1 + ab^2)$, however, we have to accept that p_0 may rise or it may fall, relative to p_n, as r rises.

It has been seen from (10) that, at $r = 0$, $(p_n - p_0) = (\ell_n - \ell_0)$ can be of either sign and from (11) that $\frac{d}{dr}(p_n - p_0)$ can be positive or negative. Now the conditions (10) and (11) for the signs of $(\ell_n - \ell_0)$ and of $\frac{d}{dr}(p_n - p_0)$ respectively are independent of one another. The possibility therefore arises that the *sign* of the correct depreciation charge in the new machine using process, $(p_n - p_0)$, may depend on the level of r. It follows from (4) and (5) that $p_n = p_0$ at $r = r^*$, where

$$r^* = \left[\frac{1 + a - c}{a(b-1)}\right] \tag{12}$$

It will be clear from (12) that if either $b < 1$ and $c > (1 + a)$ *or* $b > 1$ and $c < (1 + a)$ then r^* is positive, so that the sign of $(p_n - p_0)$, *i.e.* the *sign* of the correct depreciation charge, depends on the level of the profit rate. For example, if $b > 1$ and $(1 + a) > c > 1$, then efficiency is increasing; $\ell_n > \ell_0$ but $p_n < p_0$ for sufficiently high rates of profit.[9]

While little can be said *a priori* about depreciation, a definite result can be obtained in relation to the "capital charge" including both depreciation and profit. The correct capital charge per machine for the new machine using process, C_n, is given by

$$C_n = Rp_n \quad p_0 = \left[\frac{(1-c) + a R^2}{b + R}\right], \tag{13}$$

and that for the old machine using process by

$$C_0 = Rp_0 = \left[\frac{(c \quad 1)R + ab R^2}{b + R}\right]^{10} \tag{14}$$

It will be seen from (13) and (14) that in the constant efficiency case ($b = c = 1$),

$$C_n = C_0 = \left[\frac{aR^2}{1+R}\right]$$

which is the "textbook" annuity formula for the capital charge. In the falling efficiency case, however, $C_n > C_0$, while in the increasing efficiency case $C_n < C_0$. It will be seen, from (13), that C_n will be *negative* for "small" rates of profit if $c > (1 + a)$, *i.e.*, if $\ell_0 > \ell_n$. In the new machine using process, that is, the price of the corn output will not even

[9] While our analysis concentrates on the question of depreciation and not on that of the relation between the economists' rate of profit, r, and the accountants' various measures of the rate of profit, it is not difficult to extend the analysis to provide further examples of Harcourt's earlier discussion [2] of accountants' measures of profit rates. Thus, within the present model, Harcourt's equation (1) [2, p. 71] for the accountants' rate of profit based on "the average *book* value of capital for the year" becomes

$$R^* = \left(\frac{Q-S}{S}\right) = \left\{\frac{[1 + 2b + (\frac{c-1}{a})] + (1+b)r}{(1+b)+r}\right\} r.$$

It will be seen that:

(i) $R^* = (\frac{3+2r}{2+r})r$ when $b = c = 1$ (constant efficiency),

(ii) $R^* = (1+b)r$ when $c = 1 + ab^2$ ($p_n = a$, $p_0 = ab$),

(iii) $R^* \to (1+b)r$ as $r \to \infty$.

Similarly, Harcourt's equation (2) [2, p. 73] for the accountants' rate of profit based on calculating "annual depreciation by the reducing-balance method" becomes

$$R^*_{RB} = {}^4/3 \left(\frac{Q-S}{S}\right) = {}^4/3\, R^*.$$

[10] Returning to Table 1, it will be seen that C_n, C_0 can also be found by writing

$$bC_n + c = p_c$$

and

$$C_0 + 1 = p_c$$

which, together with (6), give results (13) and (14).

cover the corresponding wage bill; it does not follow, however, that the process is not worth operating. If there is a market in one-year-old machines, then the capitalist can add to his revenue from selling corn the proceeds of selling the old machine and the total revenue will not only cover the wage bill but will yield profit, at the rate r, on the initial purchase of the new machine. Alternatively, whether there is a second-hand market or not (p_0 being merely an accounting price in the latter case), the capitalist can use the old machine to produce corn and the *total* revenue, from corn produced by both new and old machines, will yield profit, at the rate r, on the operation considered as a whole.

Bearing in mind our assumption that $c > (1 - ab)$, it is easy to show that C_n and C_0 are both monotonically increasing functions of the profit rate and that (C_n/C_0) rises with r if $c > 1$ but falls as r rises if $1 > c > (1 - ab)$; if $c = 1$ then $C_0 = bC_n$ for all r.

Conclusion

Marx's treatment of "value depreciation" is unsatisfactory because such depreciation is not, in general, of a straight-line nature and can, indeed, even be negative. More fundamentally, "value depreciation" can only be analysed properly if machines of different ages are treated as distinct commodities, an "older" machine being produced, jointly with "corn," by labour and a "younger" machine.

When efficiency is not constant, little can be said, *a priori*, about the behaviour of the relative price of new and old machines as the rate of profit varies. This price ratio may be greater than or less than one half when profits are zero; it may rise *or* fall *or* remain constant as the profit rate rises; and it may be less than unity for some profit rates but greater than unity for others, implying that even the sign of the correct depreciation charge is not known *a priori.* Rather more definite results can, however, be obtained with respect to the capital charges, which include profits as well as depreciation charges.

REFERENCES

1 D. G. Champernowne and R. F. Kahn, "The Value of Invested Capital. A Mathematical Addendum to Mrs Robinson's Article", *Review of Economic Studies*, vol. 21, 1954.
2 G. C. Harcourt, "The Accountant in a Golden Age", *Oxford Economic Papers*, vol. 17, 1965.
3 H. R. Hudson and R. Mathews, "An Aspect of Depreciation", *Economic Record*, vol. 39, 1963.
4 K. H. Marx, *Capital*, vol. I, Moscow, 1961.
5 K. H. Marx, *Capital*, vol. II, Moscow, 1967.
6 J. von Neumann, "A Model of General Economic Equilibrium", *Review of Economic Studies*, vol. 13, 1945–46.
7 J. V. Robinson, "Depreciation", *Rivista di Politica Economica*, 1959. Reprinted in: J. V. Robinson, *Collected Economic Papers*, vol. II (Oxford, Blackwell, 1964).
8 P. Sraffa, *Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory* (Cambridge: Cambridge U.P., 1960).

Vol. 42 (1982), No. 3, pp. 281—298

Zeitschrift für
Nationalökonomie
Journal of Economics

Existence of Cost-Minimizing Systems within the Sraffa Framework

By

Neri Salvadori, Catania and Naples, Italy*

(Received July 20, 1981; revised version received February 11, 1982)

1. Introduction

During the 1960s the "choice of techniques" problem in relation to the Sraffa [18] framework was widely studied (see Morishima [11, ch. 4], Levhari [8], Garegnani [6], Łoś and Łoś [10]). Since such a study was made either inside the "Reswitching Debate" or in connection with the so-called "Non-Substitution Theorem" (see, for instance, Johansen [7], Dasgupta [3]), only single production systems were analysed (the joint production framework used in the second part of Johansen's paper [7] is not analogous to Sraffa's framework). Afterwards, some extensions were made by Schefold [16], [17]. In the former paper he analyzed truncation of fixed capital's lifetime as a choice of techniques problem (see also Baldone [1] and Varri [20]). In the latter he was concerned with "all-engaging systems", which are very particular joint production systems.

The present paper is devoted to the problem of the choice of techniques with respect to general joint production systems. Sraffa's analysis of joint production [18, part II] is formulated in terms of equations rather than inequalities. It begins by studing systems of production, each made up of a number of processes equal to the number of commodities, afterwords it studies cost-minimizing systems, i. e. systems at whose prices no operable process pays extra-profits. Thus Sraffa's analysis is completely different from von Neumann's one, even though they lead to the same results when

* I would like to thank L. Johansen and an anonymous referee of this journal for helpful comments to previous versions. Thanks are also due to C. N. R., Rome, for partial financial support. Usual caveats apply.

0044-3158/82/0042/0281/$ 03.60

only single production processes are considered. The present paper shows that the Sraffa's approach, although less studied, is as capable of general results as the von Neumann's approach.

More precisely, the paper provides a sufficient condition for the existence of cost-minimizing techniques. Among these techniques there are those which minimize the price of a particular composite commodity. Moreover, the paper considers briefly some significant cases where the above sufficient condition is easily recognized. Among these cases there are those studied by the authors previously mentioned.

In another paper [15] some difficulties of the present analysis from an economic viewpoint are analyzed and an exegetic evaluation of the definitions used in this paper is given. Here, we are mainly concerned with the mathematical framework.

2. Preliminaries and Basic Definitions

Let us assume that a finite number of commodities exist. Let n equal the number of existing commodities and let us name each commodity with a natural number from 1 to n. To simplify notation, set $N=\{1, 2, \ldots, n\}$.

A *method of production* is defined by a triplet (a, b, l), where $a \in \mathbb{R}^n$ is a vector whose elements $a_1, a_2, \ldots, a_n$ are the amounts of commodities $1, 2, \ldots, n$ which, jointly to the amount of labour $l \in \mathbb{R}$, produce the amounts of the same commodities $b_1, b_2, \ldots, b_n$ which are the elements of vector $b \in \mathbb{R}^n$. Of course, $a \geq 0$, $b \geq 0$, $l \geq 0$. The following conventions for vector inequalities are used: $x \geqq y$ if $(\forall i)$ $x^T e_i \geq y^T e_i$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $(\forall i)$ $x^T e_i > y^T e_i$, where e_i is the i-th unit vector. If $x \geqq 0$ $(x \geq 0$, $x > 0)$, x will be said *non-negative* (*semipositive, positive*).

Let us assume constant returns to scale. As a matter of fact, Sraffa himself, as I. Steedman [18] has recently pointed out, accepts such an assumption when dealing in the third part of his book with choice of techniques, even though the question of returns is definitely irrelevant in the first two parts of the book. Then, if (a, b, l) is such that $l > 0$, then $\left(\frac{1}{l}\, a, \frac{1}{l}\, b, 1\right)$ is the same method at *unitary level*. On the contrary, if $l=0$, the method (a, b, l) at *unitary level* is $\left(\frac{1}{e^T b}\, a, \frac{1}{e^T b}\, b, 0\right)$, where e is the sum vector, i. e., $e=(1, 1, \ldots, 1)^T$. In this paper, every method will always be supposed at unitary level.

Let (a_i, b_i, l_i), $i \in N$, be n methods of production. The triplet (A, B, l), where

$$A = (a_1, a_2, \ldots, a_n)^T$$
$$B = (b_1, b_2, \ldots, b_n)^T$$
$$l = (l_1, l_2, \ldots, l_n)^T$$

will be called an *n-set of methods of production* (or, more simply, an *n-set*).

If there exist a scalar r and a vector $p \in \mathbb{R}^n$ such that

$$[B-(1+r)\,A]\,p = l, \tag{2.1}$$

p will be said to be a price-vector of the *n-set* (A, B, l) at the rate of profit r. Therefore, the set

$$P\,(r) = \{p/p \in \mathbb{R}^n,\ [B-(1+r)\,A]\,p = l\}$$

will be called the *set of the price vectors* of the n-set (A, B, l) at the rate of profit r.

In a passage (Sraffa [18], p. 78; see also pp. 43—44), Sraffa could be supposed to define as "system of production" what was here defined as an "n-set". This interpretation, however, is false. In fact, dealing with single production systems, Sraffa assumes that each method produces a commodity different from commodities produced by the other methods of the same system; therefore, something else is required.

In the opinion of the author of this paper — and this point is largely discussed in the previously quoted paper [15] — a system of production can be defined only with respect to given "requirements for use". Sraffa himself, in my opinion, did not emphasize this point because in the second part of his book, where joint production systems are introduced, he dealt with *one* system, that actually in use at a single instant of time (cf. Roncaglia [13] for a similar interpretation) so that satisfaction of certain "requirements for use" are *implicitly* assumed. Furthermore, "requirements for use" are *explicitly* mentioned by Sraffa in a footnote on p. 43.

Since this paper deals with choice of techniques and since it should be meaningless to discard a system which satisfies requirements for use of a given society to obtain another which does not, "requirements for use" will be *explicitly* mentioned in the definition of the system of production.

Let $d \in \mathbb{R}^n$ be some requirements for use. Then, if there exists a vector $q \in \mathbb{R}^n$ such that

$$q^T\,(B-A) = d^T,\quad q \geq 0 \tag{2.2}$$

the n-set (A, B, l) will be called a *system of production* (or a *technique*) *with respect to the requirement for use d*. The set

$$Q(d) = \{q/q \in \mathbb{R}^n,\ q^T (B-A) = d^T,\ q \geq 0\}$$

will be called the *set of intensities* of system (A, B, l). If for some $q \in Q(d)$, $q^T e_i > 0$ $(i \in N)$, we will say that the i-th method of system (A, B, l) is *operated*.

A more exegetic derivation of the previous definitions from the writings of Sraffa may be found in the previously quoted paper [15]. Here, only three points will be stressed.

Firstly, vector d does not need to be constant, and it can be mapped by r, q, p, A, B, l. Let us give an example; if there exist a scalar g and a semi-positive vector $c \in \mathbb{R}^n$ such that

$$d^T = g\, q^T A + c^T,$$

then the requirements for use d are those of an economy where investment is carried on at the uniform growth rate g and commodities are always consumed in the same proportions (c), whatever prices and distribution are.

Secondly, the fact that all n existing commodities are required does not imply loss of generality since it is possible that only i methods involving j commodities are operated; of course $i \leq j \leq n$.

Finally, the previous definition of system does not assume that prices are non-negative. This is so because in Sraffa's opinion [18, p. 59] the positiveness of prices is obtained through competition, i. e., through choice of techniques. One of the purposes of this paper is to provide a logical framework capable of either verifying or falsifying Sraffa's opinion. We will come back again to this point in the concluding remark. A full treatment of this point can be found in Salvadori [15].

Let (a, b, l) be a method of production; if the vector $p \in \mathbb{R}^n$ and the scalar r are such that

$$[b-(1+r)\, a]^T p > l,$$

then the method (a, b, l) *pays extra profits* at prices p and at rate of profit r. On the contrary, if

$$[b-(1+r)\, a]^T p < l,$$

then the method (a, b, l) *requires extra costs* at prices p and at rate of profit r.

Let $T_h \equiv (A_h, B_h, l_h)$ and $T_k \equiv (A_k, B_k, l_k)$ be two systems of production with respect to requirements for use d, whose operated methods neither pay extraprofits nor require extra costs at the rate of profit r and at the prices of both systems T_h and T_k ($\forall\, p_h \in P_h(r)$, $\forall\, p_k \in P_k(r)$), i. e., T_h and T_k are such that

$$q_h^M [B_h - (1+r) A_h]\, p_k = q_h^M l_h \quad \forall\, p_k \in P_k(r),\ \forall\, q_h \in Q_h(d)$$

$$q_k^M [B_k - (1+r) A_k]\, p_h = q_k^M l_k \quad \forall\, p_h \in P_h(r),\ \forall\, q_k \in Q_k(d)$$

where v^M is the diagonal matrix having the elements of vector $v \in \mathbb{R}^n$ as principal diagonal elements. Then systems T_h and T_k are *compatible* at rate of profit r.

Let M be the set of the existing methods. Let the power of M be greater than the power of N, i. e., if M is finite, the existing methods are more than n. Let us assume that there exists a system of production $T_k \equiv (A_k, B_k, l_k)$ whose prices at rate of profit r, $\forall\, p_k \in P_k(r)$, are such that either no known method pays extra profits or, if method (a, b, l) does, all systems consisting of method (a, b, l) and $n-1$ methods of system T_k are compatible with it. Then, the system T_k will be called a *cost-minimizing system* at rate of profit r.

Apart from the finite number of existing commodities and constant returns to scale assumptions we will hold the following:

Labour is indispensable for the reproduction of commodities, i. e., for each n-set (A_h, B_h, l_h) there does not exist any semi-positive vector $z \in \mathbb{R}^n$ such that (2.3)

$$z^T (B_h - A_h) \geq 0 \quad \text{and} \quad z^T l_h = 0.$$

The price-vectors' set of each system of production is not empty. (2.4)

If assumption (2.4) does not hold, there exists a system of production such that the wage paid to workers cannot be different from zero. The problem of choice of techniques among systems which allow to pay a wage rate equal to zero has been rarely studied. As far as I know only M. Lippi [9] studied this problem and only for the single production case.

3. The Main Theorem

One of the aims of this paper is to prove the following theorem:

Theorem 3.1: Let M be the set of existing methods of production. Let Z be the set of all the n-sets made up of the methods in

M and let $S \subseteq Z$, $S \neq \phi$, be the set of all the n-sets which are systems of production at the requirements for use d, i. e.,

$$T_h \equiv (A_h, B_h, l_h) \in S \Leftrightarrow \exists\, q_h \in \mathbb{R}^n : q_h^T (B_h - A_h) = d^T,\ q_h \geq 0$$

d being a given function of r, q_h, p_h, A_h, B_h, l_h and defining the requirements for use of the given society. Suppose that:

(i) There exists a vector $u \in \mathbb{R}^n$ such that the set S is equal to the set of the n-sets which would be systems if the requirements for use (call them $\hat{d}$) were those of an economy where investiment is carried on at an uniform growth rate equal to the profit rate and commodities are consumed proportionally to the elements of vector u, i. e.,

$$T_h \in S \Leftrightarrow \exists\, x_h \in \mathbb{R}^n : x_h^T [B_h - A_h] = u^T + r x_h^T A_h,\ x_h \geq 0.$$

(ii) In each system the methods operated to produce the requirements for use d are also operated to produce the requirements for use $\hat{d}$, i. e.,

$$\exists\, \lambda \in \mathbb{R} : \lambda x_h \geq q_h.$$

(iii) There exists the minimum of the set

$$V = \{v / v = u^T p_h,\ p_h \in P_h(r),\ T_h \in S\}.$$

Then there exist cost-minimizing systems at rate of profit r. In particular the systems which minimize the set V are costminimizing and are compatible with each other.

Remark. Assumption (iii) of the theorem is needless when only a finite number of methods of production exists (as generally met in von Neumann-type models). Assumptions (i) and (ii) obviously hold if investment is supposed to be carried on at an uniform growth rate equal to the profit rate, and commodities are supposed to be consumed in given proportions whatever prices and distribution are, thus $d = \hat{d}$ and $q_h = x_h$, $\forall\, T_h \in S$, (such conditions are often met in von Neumann-type models, see for instance Burmeister and Kuga [2], Fujimoto [4]).

Theorem 3.1 will be proved in section 5. In this section a numerical example and some applications of the theorem will be provided.

Example. Let us consider a two commodity-three process economy whose input-output conditions are defined by Table 1. Requirements for use are defined by:

$$d = \frac{q_h^T l_h}{e_2^T p_h} e_2 + r \frac{q_h^T A_h p_h}{e_1^T p_h} e_1, \quad h \in Z$$

i. e., the economy is stationary, workers consume only commodity 2, and capitalists consume only commodity 1.

Table 1

	Inputs				Outputs	
	Commodity 1	Commodity 2	Labour		Commodity 1	Commodity 2
Process (1)	0.5	0.5	1	→	1	2
Process (2)	—	3	1	→	3	—
Process (3)	—	0.5	1	→	1	—

It is easily recognized by calculation that (a) if $0 \le r < (\sqrt{33} - 3)/4$, no system of production exists; (b) if $(\sqrt{33} - 3)/4 \le r < (\sqrt{41} - 5)/2$, there exists one system of production (made up by methods (1) and (2)), it is cost-minimizing, but assumption (i) of Theorem 3.1 does not hold; (c) if $(\sqrt{41} - 5)/2 \le r < 2\sqrt{2} - 1$, there exist two systems of production (made up by methods (1) and (2), and (1) and (3) respectively), moreover vector $u = (1, 0)^T$ fulfils the assumptions (i) and (ii) of Theorem 3.1.

Let us conclude this section by pointing out the relevance of Theorem 3.1 to same cases which are well known in the literature.

r-all-productive systems. A system (A, B, l) such that for every semipositive vector $v \in \mathbb{R}^n$ there exists a vector $z \in \mathbb{R}^n$ such that

$$z^T [B - (1+r) A] = v^T, \; z \ge 0$$

is called an r-all-productive system (see Schefold [17]). If all the existing systems are r-all-productive and their set is finite or is, however, a compact with respect to some topology defined on it, then Theorem 3.1 applies, provided that vector u is semi-positive and has so few zeros that the requirements for use $\hat{d}$ (see assumption (i) in Theorem 3.1) operate all the methods operated to produce the given requirements for use. Moreover, price-vectors of r-all-productive systems are semipositive.

r-all-engaging-systems. The r-all-productive systems whose methods are all required to obtain a net product of any commodity are called r-all-engaging systems. The r-all-engaging systems have been fully analyzed by Schefold [17] who also studied the choice of techniques between a pair of r-all-engaging systems.

Single production systems. The system (A, B, l) is a single production system if the matrix B can be transformed into a diagonal matrix with strictly positive elements along the principal diagonal by exchanges of rows (or of colums). If the Perron-Frobenius root of the non-negative matrix $B^{-1} A$ (note that in this case B^{-1} exists and is non-negative) is less than $1/(1+r)$, then the system (A, B, l) is r-all productive; moreover, if matrix A is also indecomposable, it is also r-all-engaging and is called a *basic* single production system.

r-s-partially-all-productive systems. A system (A, B, l) such that for every non-negative vector $v \in \mathbb{R}^s$, $s \in N$, there exists a vector $z \in \mathbb{R}^n$ such that

$$z^T [B-(1+r) A] = (v^T, 0^T), \; z \geq 0,$$

is called an r-s-partially-all-productive system. If all existing systems are r-s-partially-all-productive and their set is finite or is, however, a compact with respect to some topology defined on it, then Theorem 3.1 applies provided that the first s elements of vector u are positive, all others being zero. Moreover, the prices of the first s commodities are non-negative.

Fixed capital systems. Fixed capital systems, as defined by Schefold [16], are r-s-partially-all-productive systems (if $r \in [0, R[$, where R is the maximum rate of profit), the finished goods being the first s commodities and the intermediate goods being the second $(n-s)$ commodities. Furthermore, if life-time truncation of fixed capital goods is allowed (i. e., if for every method producing a final good and an intermediate good there exists another method having the same inputs and the same outputs except that the intermediate good is not produced), then cost-minimizing systems have non-negative prices for all produced commodities (since the prices of the finished goods are non-negative within each system, use Theorem 3.1 and observe that if an intermediate good has a negative price, then a greater rate of profit can be obtained by discarding the production of that intermediate good).

Other sorts of fixed capital systems can be easily introduced. For example, systems which allow either methods utilizing jointly machines (intermediate goods) whose efficiency is constant over their lifetime, or methods producing some scrap fully utilized in production. Fixed capital systems of these sorts (without joint production overimposed), are s-r-partially-all productive (if $r \in [0, R[$). Moreover, if lifetime truncation is allowed, cost-minimizing systems admit non-negative prices for all produced commodities.

4. Preferability between Systems

In this section an auxiliary concept will be introduced and fully analyzed. In the next section this concept will be utilized to prove the main theorem and some related results. Since the rate of profit will be assumed as given at level r throughout the paper, let us simplify notation by setting

$$C = B - (1+r)\,A.$$

Assume that $T_h = (A_h, B_h, l_h)$ and $T_k = (A_k, B_k, l_k)$ are two systems which have $n-1$ common methods, but one is different. If, at the rate of profit r,

$$\begin{aligned} C_h\, p_k &\leq l_h, \quad \forall\, p_k \in P_k\,(r), \\ C_k\, p_h &\geq l_k, \quad \forall\, p_h \in P_h\,(r), \end{aligned} \tag{4.1}$$

then system T_k will be said to be *preferable* to system T_h at rate of profit r.

Let us remark that even though T_k is preferable to T_h, they can be compatible as well if the non-common methods are not operated in both systems. Moreover, if T_h and T_k are not compatible and the $n+1$ methods they are made up of are all the existing methods, then T_k is preferable to T_h if and only if T_k is a cost-minimizing system and T_h is not.

This section is mainly concerned to prove the following theorems:

Theorem 4.1: If the system (A_k, B_k, l_k) is preferable to system (A_h, B_h, l_h) at rate of profit r, then

$$\text{rank } C_h = \text{rank } C_k = \text{rank } (C_h{}^T, C_k{}^T).$$

Proof: Otherwise $P_h\,(r) \cap P_k\,(r) \neq \emptyset$.

Theorem 4.2: Let (A_h, B_h, l_h) and (A_k, B_k, l_k) be two systems consisting of the same methods but the s-th which is different. The price vectors at rate of profit r relative to the two systems are not equal, i. e. $P_h\,(r) \cap P_k\,(r) = \emptyset$.

Then the following statements are equivalent:

(a) One of the two systems is preferable to the other at rate of profit r.

(b) $\exists\, v, x_h, x_k \in \mathbb{R}^n$:

$$x_h{}^T\, C_h = x_k{}^T\, C_k = v^T, \quad x_h > 0, \quad x_k > 0.$$

(c) Let $U = \{u / u \in \mathbb{R}^n,\; u^T p_h \neq u^T p_k,\; \forall\, p_h \in P_h\,(r),\; \forall\, p_k \in P_k\,(r)\}$, and

$$Z_i(u) = \{z/z \in \mathbb{R}^n,\ z^T C_i = u^T\} \quad (i = h, k;\ u \in U);$$

then

$$z_h^T e_s z_k^T e_s > 0$$

$$\forall z_h \in Z_h(u),\ \forall z_k \in Z_k(u), \text{ and } u \in U.$$

Moreover, if statements (a), (b), (c) hold, then (A_k, B_k, l_k) is preferable to (A_h, B_h, l_h) if and only if

$$u^T p_k < u^T p_h \Leftrightarrow z_i^T e_s > 0 \quad (z_i \in Z_i(u);\ i = h, k).$$

For the proof of Theorem 4.2 we need the following Lemma which is a well known Theorem of the Alternative (see Gale [5], p. 49).

Lemma 1: Either the equation

$$x^T A = 0^T$$

has a positive solution, or the inequality

$$Ay \geq 0$$

has a solution. But never both.

Proof of Theorem 4.2. Assume that statement (b) holds. Let $y \in \mathbb{R}^n$ be a vector such that

$$C_k y \leq 0, \quad \hat{C}y = 0 \tag{4.2}$$

where $\hat{C}$ is the $(n-1)\,xn$ matrix obtained from either C_h or C_k by dropping the s-th row. Then, by Lemma 1 there exists no vector $z \in \mathbb{R}^n$ such that $(C_k^T, -C_h^T)^T z \leq 0$. Thus, since rank C_h = rank C_k = rank (C_h^T, C_k^T),

$$C_h y \leq 0, \tag{4.3}$$

which will be used below.

Since

$$\hat{C} p_i = \hat{l}, \quad (i = h, k)$$

where $\hat{l} \in \mathbb{R}^{n-1}$ is the vector obtained from either l_h or l_k by dropping the s-th element, for every $p_h \in P_h(r)$ and $p_k \in P_k(r)$ there exist $m+1$ real numbers $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$p_h - p_k = \lambda y + \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_m y_m \tag{4.4}$$

where $m = n - \text{rank } C_h$ and $y_1, y_2, \ldots, y_m$ are m linearly independent solutions of the vector-equation

$$C_h w = 0.$$

Note that $\lambda \neq 0$, since $P_h(r) \cap P_k(r) = \emptyset$ by hypothesis. Hence, by (4.4),

$$C_h p_k = C_h p_h - \lambda C_h y = l_h - \lambda C_h y. \tag{4.5}$$

On the other hand, y_i $(i = 1, 2, \ldots, m)$ is also a solution of the equation

$$C_k w = 0,$$

otherwise $P_h(r) \cap P_k(r) \neq \emptyset$. Therefore,

$$C_k p_h = C_k p_k + \lambda C_k y = l_k + \lambda C_k y. \tag{4.6}$$

Thus, since (4.2) and (4.3) hold, (4.5) and (4.6) imply that

$$C_h p_k - l_h \geq 0 \Leftrightarrow C_k p_h - l_k \leq 0,$$

which proves that (b) $\Rightarrow$ (a).

Assume now that $z_h \in Z_h(u)$ and $z_k \in Z_k(u)$ $(u \in U)$ and obtain, from (4.5) and (4.6), that

$$\begin{aligned} u^T p_k &= u^T p_h - \lambda z_h^T C_h y, \\ u^T p_h &= u^T p_k + \lambda z_k^T C_k y, \end{aligned} \tag{4.7}$$

i. e.

$$\lambda = \frac{u^T p_h - u^T p_k}{z_h^T C_h y} = \frac{u^T p_h - u^T p_k}{z_k^T C_k y}.$$

Therefore, (b) $\Rightarrow$ (c) since all elements of vector $C_i y$ $(i = h, k)$ are zero except the s-th which is negative.

By the same argument, the second part of the theorem is also proved since (4.5) and (4.6) can be rewritten

$$C_h p_k - l_h = -\frac{u^T p_h - u^T p_k}{z_i^T C_i y} C_h y \quad (i = h, k)$$

$$C_k p_h - l_k = \frac{u^T p_h - u^T p_k}{z_i^T C_i y} C_k y \quad (i = h, k)$$

To complete the proof we need to prove (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b). This will be done by proving that if (b) does not hold, then neither (a) nor (c) hold.

Assume that (b) does not hold. Then, because of Lemma 1, there exists a vector $y \in \mathbb{R}^n$ such that $(C_h^T, -C_k^T)^T\, y \leq 0$. Thus, since rank C_h = rank C_k = rank (C_h^T, C_k^T),

$$C_h\, y \leq 0,\ C_k\, y \geq 0. \tag{4.8}$$

Therefore,

$$\hat{C} y = 0,\ y \neq 0.$$

Hence (4.4)—(4.7) hold. But, because of (4.8),

$$C_h\, p_k - l_h \geq 0 \Leftrightarrow C_k\, p_h - l_k \geq 0$$

and

$$(z_h^T\, e_s)\,(z_k^T\, e_s) < 0.$$

Q. E. D.

Corollary: In the hypothesis of Theorem 4.2, if $|C_i| \neq 0$ $(i=h, k)$, then one of the two systems is preferable to the other if and only if

$$|C_h| \cdot |C_k| > 0.$$

Proof: It is sufficient to remark that $|C_h| \cdot |C_k| < 0$ if and only if, in space $\mathbb{R}^n$, the points $C_h^T\, e_s$ and $C_k^T\, e_s$ are contained in the two open half-spaces produced by the hyperplane which contains the origin and the points $C_h^T\, e_j$, $\forall\, j \in N - \{s\}$.

5. Proof of the Main Theorem and Related Theorems

In this section, before proving the main theorem, requirements for use will be assumed to be equal to $\hat{d}$. Moreover, let us simplify notation by setting

$$P_h = P_h\,(r),$$

$$X_h = Q_h\,(\hat{d}).$$

Thus, before proving the main theorem, the method $(A_h^T\, e_i,$ $B_h^T\, e_i,\, l_h^T\, e_i)$ of system T_h will be considered as operated if and only if there exists a vector $x \in X_h$ such that $x^T\, e_i > 0$. Finally, because of assumption (2.4),

$$x_h^T\, l_h = u^T\, p_h,\ \forall\, x_h \in X_h,\ \forall\, p_h \in P_h. \tag{5.1}$$

Theorem 5.1: Let $T_h \equiv (A_h, B_h, l_h)$ be a system of production and let (a, b, l) be a method of production such that $b-(1+r)\, a = c$ is not a linear combination of the rows of $C_h = B_h - (1+r)\, A_h$. Then,

(1) there does not exist any system consisting of the method (a, b, l) and $n-1$ methods of system T_h such that the method (a, b, l) is operated;
(2) there exists a system $T_k \equiv (A_k, B_k, l_k)$ consisting of the method (a, b, l) and $n-1$ methods of system T_h such that T_k is compatible with T_h;
(3) for every system T_k as defined in (2), $u^T p_h = u^T p_k$;
(4) for every system T_k as defined in (2), rank $C_k = 1 +$ rank C_h.

Proof: Let $\hat{w} \in \mathbb{R}^{n+1}$ be a non-zero solution of the vector-equation

$$w^T (C_h^T, c)^T = 0^T,$$

then $\hat{w}^T e_{n+1} = 0$. Therefore, part (1) is proved since all solutions of vector-equation in $(v^T, \alpha)^T$

$$(v^T, \alpha) (C_h^T, c)^T = u^T$$

are

$$(x_h^T, 0)^T + \lambda \hat{w}, \ \forall \ x_h \in X_h, \ \forall \ \lambda \in \mathbb{R}.$$

To prove part (2), assume, without loss of generality, that $x_h^T e_i = 0$ for some $i \in N$. Then, the system T_k consisting of all the methods of system T_h except the method i-th which is substituted by method (a, b, l) is compatible with T_h, since $P_h \supset P_k$. Proofs of parts (3) and (4) are trivial. Q. E. D.

Theorem 5.2: Let $T_h \equiv (A_h, B_h, l_h)$ be a system of production and let (a, b, l) be a method of production such that:

(a) $b - (1 + r)\, a = c$ is a linear combination of the rows of C_h, i. e., $\exists \ y \in \mathbb{R}^n : c^T = y^T C_h$,

(b) $c^T p_h = y^T l_h > l$,

(c) $\exists \ j \in \{s / s \in N, \ x_h^T e_s = 0, \ \forall \ x_h \in X_h\}$: $y^T e_j > 0$.

Then:
(1) as (1) in Theorem 5.1,
(2) as (2) in Theorem 5.1,
(3) as (3) in Theorem 5.1,
(4) if $|C_h| \neq 0$, then T_k, as defined in (2); is preferable to T_h.

Proof: The general solution of the vector equation

$$(v^T, \alpha) (C_h^T, c)^T = u^T$$

is

$$(\hat{v}^T, \hat{\alpha}) = (x_h^T, 0) + \hat{\alpha} (-y^T, 1), \ \forall \ x_h \in X_h, \ \forall \ \hat{\alpha} \in \mathbb{R}.$$

Therefore, because of hypothesis (c),

$$\hat{\alpha} > 0 \Rightarrow (\hat{v}^T, \hat{\alpha})^T \ngeq 0,$$

and this proves part (1). Then, because of (1) and (c), the system (A_k, B_k, l_k) consisting of all the methods of T_h but the j-th which is substituted by method (a, b, l) is compatible with T_h. Hence part (2) is proved. Proof of part (3) is trivial. To prove part (4) it is sufficient to remark that

$$(|C_k|/|C_h|) = c^T C_h^{-1} e_j = y^T e_j > 0.$$

Q. E. D.

Theorem 5.3: Let $T_h \equiv (A_h, B_h, l_h)$ be a system of production and let (a, b, l) be a method of production such that:

(a) as (a) in Theorem 5.2;

(b) as (b) in Theorem 5.2;

(c) $\forall\, j \in \{s/s \in N,\ x_h e_s = 0,\ \forall\, x_h \in X_h\},\ y^T e_j \leq 0$.

Then:

(1) There exists a system of production $T_k = (A_k, B_k, l_k)$ preferable to T_h and made up of the method (a, b, l) and of $n-1$ methods of system T_h.

(2) For every system T_k as defined in (1):
  (i) the method (a, b, l) is operated with system T_k;
  (ii) $u^T p_h > u^T p_k$.

For the proof of Theorem 5.3 we need the following Lemma which is a known Theorem of the Alternative (see Gale [5], p. 44).

Lemma 2: Either the equation

$$x^T A = b^T$$

has a non-negative solution or the inequalities

$$A y \geqq 0,\ b^T y < 0$$

have a solution. But never both.

Proof of Theorem 5.3. Let $\hat{x}_h \in X_h$ have as many positive elements as possible. The vectors

$$(w^T, \lambda)^T = (\hat{x}_h{}^T, 0)^T + \lambda\, (-y^T, 1)^T, \quad \forall\, \lambda \in \mathbb{R}$$

are solutions of the vector-equation

$$(v^T, \alpha)\ (C_h{}^T, c)^T = u^T.$$

Then, hypothesis (c) implies that

— there exists some $\lambda > 0$ such that $\hat{x}_h - \lambda y \geq 0$.
— if $\bar{\lambda} > 0$ and $\hat{x}_h - \bar{\lambda} y \geq 0$, then $\hat{x}_h - \lambda y \geq 0$, $\forall\, \lambda \in [0, \bar{\lambda}]$.

Moreover, since $C_h\, p_h = l_h$ and $l_h \geq 0$ because of assumption (2.3), y cannot be non-positive because of hypothesis (b) and Lemma 2. Therefore, it is recognized that there exists a value $\hat{\lambda}$ of λ such that

$$\hat{\lambda} > 0 \tag{5.2}$$

$$\hat{w} \equiv \hat{x}_h - \hat{\lambda} y \geq 0 \tag{5.3}$$

$$\exists\, i \in \{s/s \in N,\ \hat{x}_h^T\, e_s > 0\};\ \hat{w}^T\, e_i = 0. \tag{5.4}$$

Then system $T_k = (A_k, B_k, l_k)$, which consists of all the methods of system T_h but the i-th method which is substituted by method (a, b, l) is preferable to T_h. In fact, statement (c) of Theorem 4.2 holds because of inequalities (5.2) and (5.4), and inequality

$$C_k\, p_h - l_k \geq 0 \tag{5.5}$$

holds by hypothesis (b). Hence parts (1) and (2) (i) of the theorem are proved. To prove (2) (ii), it is sufficient to remark that vectors x_k and $C_k\, p_h - l_k$ are not orthogonal, and that inequality (5.5) and Eq. (5.1) hold. Q. E. D.

Theorem 5.4: Let $T_h \equiv (A_h, B_h, l_h)$ and $T_k \equiv (A_k, B_k, l_k)$ be two systems such that

$$u^T\, p_h > u^T\, p_k.$$

Then:

(1) There exists a method of system T_k which pays extra profits at prices of system T_h;
(2) Systems T_h and T_k are not compatible.

Proof: Assume that (1) does not hold. Then, for some $p_h \in P_h$,

$$C_k\, p_h \leqq l_k,$$

therefore,

$$u^T\, p_h = x_k^T\, C_k\, p_h \leq x_k^T\, l_k = u^T\, p_k,$$

where $x_k \in X_k$. Hence, a contradiction, and (1) holds. If T_k and T_h are compatible, then the vectors $C_h\, p_k - l_h$ and x_h ($p_k \in P_k$, $x_h \in X_h$) are orthogonal and thus

$$u^T\, p_k = x_h\, C_h\, p_k = x_h\, l_h = u^T\, p_h.$$

Hence, a contradiction and (2) holds. Q. E. D.

Proof of the main theorem. Let us begin by proving that if the requirements for use are $\hat{d}$, a system is cost-minimizing if and only if it minimizes the set V; if more than one such a system exist, they are compatible with each other. To do this, notice that, as a consequence of Theorem 5.4, systems which do not minimize $u^T p_h$ ($T_h \in Z$) cannot be cost-minimizing. Then, let T_k be a system such that

$$u^T p_k \leq u^T p_h \quad \forall\, T_h \in Z \tag{5.6}$$

and assume that there exists a method (a, b, l) which pays extra profits at prices p_k. Obviously T_k and (a, b, l) cannot fulfil the hypotheses of Theorem 5.3, otherwise $u^T p_k$ could not fulfil inequality (5.6). Therefore by Theorems 5.1 and 5.2 T_k is a cost-minimizing system and all systems for which $u^T p_h$ ($T_h \in Z$) is a minimum are compatible with each other.

To complete the proof it is sufficient to remark that because of assumption (i) and (ii) the sets of systems of production defined by requirements for use d and $\hat{d}$ coincide, and that the systems which are compatible with respect to requirements for use $\hat{d}$ are also compatible with respect to requirements d.

Q. E. D.

Remark. If $d \neq \hat{d}$, the set of cost-minimizing systems may strictly include the set of the systems which minimize the set V. This fact occurs even if only single product systems are involved. Let us give an example: Assume that commodity j is not produced within all cost-minimizing systems (thus, it is non-basic in all of them but basic in some other system), and $u^T e_j > 0$.

The following theorem can be useful.

Theorem 4.5: In the main theorem, assumption (iii) can be replaced by the following assumption:

(iv) There exists a topology τ such that the set S is a compact with respect to τ.

Proof. It is sufficient to prove that in the other assumptions of the main theorem (iv) $\Rightarrow$ (iii). By Theorems 4.1, 4.2, 4.3, 4.4 it is possible to construct a sequence $\{T_{h_t}\}$ such that the sequence $\{u^T p_{h_t}\}$ is non-increasing and convergent to inf V. Since S is a compact, $\{T_{h_t}\}$ contains a convergent subsequence whose limit, T_k, is in S, thus

$$u^T p_k = \min \mathrm{V}. \quad \forall\, p_k \in P_k(r).$$

Q. E. D.

6. Concluding Remark

This paper provides a sufficient condition for the existence of cost-minimizing systems within the Sraffa framework. However, it does not prove that:

(i) if some systems of production exist, there exists a costminimizing system;
(ii) if a cost-minimizing system exists, and if a system with a non-negative price-vector exists, then there exists a cost-minimizing system with a non-negative price-vector.

These propositions can be proved in particular cases. But they cannot be proved in the general, joint production case: examples are provided by the paper quoted in the introduction ([15], see also [14]). In the opinion of the present author, these missing proofs are strong difficulties for the Sraffa joint production framework. The quoted paper [15] is mainly devoted to working out a solution by discarding Sraffa's own assumption setting the number of methods of production equal, in each system, to the number of commodities involved. In this way, however, some differences between the Sraffa approach and the von Neumann approach have been made to vanish.

References

[1] S. Baldone: Fixed Capital in Sraffa's Theoretical Scheme, in: L. L. Pasinetti (ed.): Essays on the Theory of Joint Production, London 1980, pp. 88—137.

[2] E. Burmeister and K. Kuga: The Factor-Price Frontier, Duality and Joint Production, Review of Economic Studies *37* (1970), pp. 11—19.

[3] G. Dasgupta: A Note on Johansen's Non-Substitution Theorem and Malinvaud's Decentralization Theorem Procedure, Journal of Economic Theory *9* (1974), pp. 340—349.

[4] T. Fujimoto: Duality and the Uniqueness of Growth Equilibrium, International Economic Review *16* (1975), pp. 781—791.

[5] D. Gale: The Theory of Linear Economic Models, New York etc. 1960.

[6] P. Garegnani: Nota Matemetica; published as appendix to: Beni Capitali Eterogenei. La Funzione della Produzione e la Teoria della Distribuzione, in: P. Sylos Labini (ed.): Prezzi Relativi e Distribuzione del Reddito, Turin 1973, pp. 332—345. This mathematical note was mentioned but not published in the original, English edition of the paper: Review of Economic Studies *37* (1970), pp. 407—436.

[7] L. Johansen: Simple and General Non-Substitution Theorems for Input-Output Models, Journal of Economic Theory *5* (1972), pp. 383—394.

[8] D. Levhari: A Nonsubstitution Theorem and Switching of Techniques, Quarterly Journal of Economics *79* (1965), pp. 98—105.

[9] M. Lippi: I Prezzi di Produzione. Un Saggio sulla Teoria di Sraffa, Bologna 1979.

[10] J. Łoś and M. W. Łoś: Reswitching of Techniques and Equilibria of Extended von Neumann Models, in: M. W. Łoś, J. Łoś and A. Wieczorek (eds.): Warsaw Fall Seminars in Mathematical Economics 1975, Berlin – Heidelberg – New York 1976, pp. 97—118.

[11] M. Morishima: Equilibrium Stability, and Growth. A Multi-Sectoral Analysis, Oxford 1964.

[12] J. von Neumann: Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, Ergebnisse eines mathematischen Kolloquiums *8* (1936), pp. 73—83. English translation: A Model of General Economic Equilibrium, Review of Economic Studies *13* (1945/46), pp. 1—9.

[13] A. Roncaglia: Sraffa and the Theory of Prices, New York 1978.

[14] N. Salvadori: Mutamento dei Metodi di Produzione e Produzione Congiunta, Quaderni dell'Istituto di Economia dell'Universita di Siena Nr. 6, Siena 1979.

[15] N. Salvadori: Switching in Methods of Production and Joint Production, unpublished mimeo, Napoli 1981.

[16] B. Schefold: Fixed Capital as a Joint Product, Jahrbücher für Nationalökonomie und Statistik *192* (1977), pp. 415—439.

[17] B. Schefold: Multiple Product Techniques with Properties of Single Product Systems, Zeitschrift für Nationalökonomie *38* (1978), pp. 29—53.

[18] P. Sraffa: Production of Commodities by Means of Commodities, Cambridge 1960.

[19] I. Steedman: Returns to Scale and the Switch in Methods of Production, Studi Economici *35* (1980), pp. 5—14.

[20] P. Varri: Prices, Rate of Profit and Life of Machines in Sraffa's Fixed Capital Model, in: L. L. Pasinetti (ed.): Essays on the Theory of Joint Production, London 1980, pp. 55—87.

Address of author: Prof. Neri Salvadori, Istituto di Matematica, Facoltà di Economia e Commercio, Via Partenope 36, I- 80121 Napoli, Italy.

[5]

SWITCHING IN METHODS OF PRODUCTION AND JOINT PRODUCTION*

by
NERI SALVADORI†
University of Catania

> "The skins of the larger animals were the original materials of clothing. Among nations of hunters and shepherds, therefore, whose food consists chiefly in the flesh of those animals, every man, by providing himself with food, provides himself with the materials of *more clothing than he can wear*. If there was no foreign commerce, the greater part of them would be *thrown away as things of no value*. This was probably the case among the hunting nations of North America, before their country was discovered by the Europeans, with whom they now exchange their surplus peltry, for blankets, fire-arms, and brandy, which gives it *some value*."
>
> (Adam Smith, *The Wealth of Nations*, Book I, Chapter 11, emphasis added. Quoted by H. D. Kurz, 1980.)

I Introduction

Sraffa deals with the problem of "choice of techniques" in Chapter XII of *Production of Commodities by Means of Commodities*. There, however, he is mainly concerned with the single product case, i.e., the particular case obtained by assuming that each industry produces one commodity. Only a few remarks on the general joint production case are provided in Section 96.

Since the publication of Sraffa's book the single production case has been fully analysed (*c.f.* Levhari, 1965; Garegnani, 1973; Morishima, 1964; Łos and Łos, 1976; Lippi, 1979), but only particular joint production cases have been investigated: see Baldone (1980); Schefold (1977); Varri (1980), who analyse the truncation of fixed capital's lifetime as a choice of techniques problem; and Schefold (1978a), who is concerned with "all-engaging systems", which are very particular joint production systems.

The present paper is devoted to the problem of the choice of techniques

*Manuscript received 25.7.83; final version received 7.3.84.

†I would like to thank the participants at the seminars in the Universities of Bremen, Frankfurt, Rome, Manchester, Berlin and the New School for Social Research (New York), where this paper was presented. Thanks are also given to the Italian MPI (Ministry of Public Education) for financial support. The usual caveats apply.

with respect to general joint production systems. More precisely, the paper tries first of all to provide the general definitions of "system of production" and "cost-minimizing system" actually utilized by Sraffa. Secondly, the paper shows that

(a) Sraffa's remarks on choice of techniques, with respect to joint production, are false;

(b) a cost-minimizing system does not need to have non-negative prices, even if some other system does;

(c) a cost-minimizing system does not need to exist, even if some system does exist.

These facts are first shown by examples and then discussed with the help of a simple, geometric exposition of the theory of choice of techniques in a three process-two commodity economy. Another, more mathematical, paper (Salvadori, 1982) provides the theory of choice of techniques in the general joint production case (with a finite number of commodities).[1]

Thirdly, the above mentioned propositions (b) and (c) are shown to be remarkable, since they are in contrast with the method of long-period positions followed by Sraffa (as Garegnani, 1976, has stated). A way to solve this problem is provided by discarding Sraffa's own assumption setting the number of methods of production equal, in each system, to the number of commodities involved. In this way, however, some differences between the Sraffa approach and the von Neumann approach have been made to vanish.

I wish to assert explicitly here that returns to scale will be assumed constant throughout the paper. As a matter of fact, Sraffa himself, as Steedman (1980) has recently pointed out, implicitly accepts such an assumption when dealing, in Chapter XII of his book, with choice of techniques, even though the question of returns is definitely irrelevant in the first eleven chapters of the book. As far as I know, the problem of choice of techniques

[1]This paper was almost finished when I received a paper by Bidard (1984), which suggests a formulation of the Sraffa joint production framework different from that provided here.

The main difference between Bidard's approach and mine has been clarified by Bidard himself (*c.f.* Bidard, 1984, pp. 206-7, who refers to Salvadori, 1982). Instead of defining the system of production as a set of n processes satisfying given requirements for use (n being the number of commodities), Bidard first divides all the existing processes among n sectors, then he defines a system of production as an n-process set made by taking one process from each sector, so that no system contains two processes which are in the same sector.

Bidard's paper provides neither exegetic references to Sraffa's writings nor economic rationale to support division of processes into sectors. Hence his contribution is not dealing with the theme we are concerned with here. However, it is highly valuable with respect to proving the existence of cost minimizing systems: it provides a sufficient condition for the existence of cost-minimizing systems which is different (neither more general, nor more particular) from that provided by Salvadori (1982).

within Sraffa's framework has never been analysed without assuming constant returns to scale. Difficulties related to such a problem are emphasized by Steedman (1980).

II A Definition of "System of Production"

In the first part of *Production of Commodities by Means of Commodities*, Sraffa does not define formally the *system of production* Thus the reader has to provide such a concept by himself. However, no difficulty in interpretation arises since, if all methods of production are single product methods and k is the number of commodities involved in production, then at least k methods must be operated; and k methods determine k constraints among the $k - 1$ relative prices, the wage rate, and the profit rate. Therefore, prices and the wage rate (profit rate) can be determined as functions of the profit rate (wage rate).

Introducing joint production (Sraffa, 1960, Section 50), Sraffa encounters a difficulty: since some method produces more than one commodity, fewer methods than the number of commodities involved could be operated. "The conditions would no longer be sufficient to determine the prices. There would be more prices to be ascertained than there are processes, and therefore equations, to determine them". Then, Sraffa suggests meeting this difficulty by an assumption. He assumes that "the number of processes should be equal to the number of commodities".

Another difficulty which arises when joint production is involved is the following: even though returns to scale are constant and the operable methods of production are equal in number to the commodities involved, such methods may not be able to fulfil whatever "requirements for use" may exist. Sraffa is conscious of this:

> "Incidentally, considering that the proportions in which the two commodities are produced by any one method will in general be different from those in which they are *required for use*, the existence of two methods of producing them in different proportions will be necessary for obtaining the required proportion of the two products through an appropriate combination of the two methods."
> (Sraffa, 1960, §50, p. 43n; italics added.)

Indeed, Sraffa is very aware of the mentioned difficulty:

> "Take for example the case of two products jointly produced by each of two different methods. The possibility of varying the extent to which one or the other method is employed ensures a certain range of variation in the proportions in which the two goods may be produced in the aggregate. But this range finds its limits in the proportions in which the

two goods are produced respectively by each of the two methods, so that the limits are reached as soon as one or the other method is exclusively employed."

(Sraffa, 1960, §53, p. 47.)

However, Sraffa does not mention this problem in defining the system of production (*c.f.* Sraffa, 1960, §51; see also the last paragraph of §90, p. 78). I think that this is because when joint production systems are introduced, Sraffa is dealing with *one* system (either that actually in use at a single instant of time, as Roncaglia, 1978, suggests, or that which has to prevail in the long run because of competition, as Garegnani, 1976, suggests). The intellectual experiment Sraffa is going to perform consists in evaluating the variations of prices and the wage rate as functions of the profit rate by assuming that there are "no changes in output and (. . . .) no changes in the proportions in which different means of production are used by an industry" (Sraffa, 1960, p. v). Therefore, the satisfaction of certain "requirements for use" is *implicitly* assumed.[2]

In this paper "requirements for use" will be *explicitly* mentioned in defining the system of production. This paper deals with choice of techniques and it would be meaningless to discard a system which satisfies requirements for use of a given society to obtain another which does not. We can now give the following definition: *a system of production à la Sraffa* is a set of processes whose number is equal to the number of commodities involved in production and such that they fulfil exactly given "requirements for use".

Let us put the matter more formally.

A *method of production* (or a *process*) is defined by a triplet $(\mathbf{a}, \mathbf{b}, l)$, where $\mathbf{a}$ is an n-vector whose elements $a_1, a_2, \ldots, a_n$ are the amounts of commodities $1, 2, \ldots, n$ which, jointly with the amount of labour l, produce the amounts of the same commodities $b_1, b_2, \ldots, b_n$ which are the elements of the n-vector $\mathbf{b}$. Of course $\mathbf{a} \geqslant 0, \mathbf{b} \geqslant 0, l \geqslant 0$.

Definition 2.1. Let the non-negative n-vector $\mathbf{d}$ be some requirements for use, and let $(\mathbf{a}_i, \mathbf{b}_i, l_i)$, $i = 1, 2, \ldots, n$, be n methods of production. Then the triplet $(A, B, \mathbf{l})$, where

$$A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)^T$$
$$B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)^T$$
$$\mathbf{l} = (l_1, l_2, \ldots, l_n)^T$$

will be called a *system of production* (or a *technique*) *with respect to the require-*

[2]Similar interpretations were provided by Roncaglia (1978) (see Chapter I, §8) and Garegnani (1984), who arranges the problems traditionally dealt with by economists in two groups, problems which are inside the "core" and problems which are outside it: prices are determined within the core, assuming produced quantities as determined outside it. Both Garegnani's and Roncaglia's remarks, however, are mainly concerned with returns to scale rather than with joint production.

ments for use d if there exists an n-vector $\mathbf{x}$ such that

$$\mathbf{x}^T (B - A) = \mathbf{d}^T, \mathbf{x} \geqslant 0 \qquad \text{......(2.1)}$$

A vector $\mathbf{x}$ satisfying constraints (2.1) will be said to be an *intensity-vector* of system $(A, B, \mathbf{1})$. If the ith element of an intensity-vector of system $(A, B, \mathbf{1})$ is positive, we will say that the ith method of system $(A, B, \mathbf{1})$ is *operated*.

It must be stressed that vector $\mathbf{d}$ does not need to be constant, and it can be a function of the profit rate, wage rate, price-vector, intensity-vector, labour vector, input matrix, and output matrix. Let us give an example; if there exists a scalar g and a semi-positive n-vector $\mathbf{c}$ such that

$$\mathbf{d}^T = g\, \mathbf{x}^T A + \mathbf{c}^T,$$

then the requirements for use $\mathbf{d}$ are those of an economy where investment is carried on at the uniform growth rate g and commodities are always consumed in the same proportions $\mathbf{c}$, whatever prices and distribution may be.

As is well known, the prices of production for each system are defined by the following equation:

$$B\,\mathbf{p} = (1 + r)\, A\mathbf{p} + w\mathbf{1} \qquad \text{......(2.2)}$$

$$\mathbf{q}^T\mathbf{p} = 1 \qquad \text{......(2.3)}$$

where $\mathbf{p}$ is the price-vector, r is the profit-rate and w is the wage rate. Note that equation (2.3) normalizes prices by setting as numéraire a composite good made up of q_1 units of commodity 1, q_2 units of commodity 2, ... , q_n units of commodity n, where $(q_1, q_2, \ldots\ldots, q_n)^T = \mathbf{q}$. Alternatively, if the numéraire is labour, the prices are defined by the following equation:

$$B\hat{\mathbf{p}} = (1 + r)\, A\hat{\mathbf{p}} + \mathbf{1}$$

where $\hat{\mathbf{p}}$ is also named the labour-commanded vector.

I want to remark that once one of the two distribution variables is exogenously given, the prices are determined by the conditions of production only. Requirements for use act exclusively in determining the set of systems of production, but after that they play no role in determining prices, which could well be called "prices of production" (see Sraffa, 1960, §7, p. 9).

III A Definition of "The Cost-Minimizing System"

As was stated above, Sraffa analyses the problem of the determination of the cost-minimizing system in Chapter XII of his book. Sraffa begins this chapter by assuming that there are $k + 1$ single product methods to produce k commodities, i.e., two alternative methods to produce one commodity are known. Therefore two different systems of production can be constructed, using either the former or the latter of the alternative methods. Then Sraffa is able to prove that if a method produces at lower costs, by buying inputs at

the prices of the former system, then it produces at lower costs also by buying inputs at the prices of the latter system and *vice versa*. Hence the concept of the cost-minimizing system is unambiguous: the cost-minimizing system is the system whose prices are such that no method can produce a commodity at lower costs. There exist three other definitions which are easily recognized to be equivalent (if only single production is allowed) to the previous one: a cost-minimizing system is a system whose prices are such that no method can
—pay extra profits,
—pay a rate of profit greater than the current one,
—produce some commodity at lower costs per unit of produced value.

These equivalents are particularly useful when joint production is introduced. In fact, in this case it is not always possible to define the cost of *one* commodity, since a method may produce more than one commodity and a commodity can be produced, in each system, by more than one method. On the other hand, "extra profits", "rate of profit greater than the current one" and "costs per unit of value" can obviously be defined even if joint production is introduced.

Again, let us put the matter in a more formal way. Let $(\mathbf{a}, \mathbf{b}, l)$ be a method of production; if the n-vector $\mathbf{p}$ and the scalars r and w are such that

$$[\mathbf{b} - (1 + r)\mathbf{a}]^T\mathbf{p} > wl$$

then the method $(\mathbf{a}, \mathbf{b}, l)$ *pays extra profits* at prices $\mathbf{p}$, at rate of profit r and at wage rate w. On the contrary, if

$$[\mathbf{b} - (1 + r)\mathbf{a}]^T\mathbf{p} < wl$$

then the method $(\mathbf{a}, \mathbf{b}, l)$ *requires extra costs* at prices $\mathbf{p}$, at rate of profit r and at wage rate w. Then:

Definition 3.1. A *cost-minimizing system* at rate of profit r is a system of production whose prices and wage rate at rate of profit r are such that no known method of production pays extra profits, i.e., system $(A_k, B_k, \mathbf{1}_k)$ is cost-minimizing at rate of profit r if and only if

$$[B_h - (1 + r)A_h]\mathbf{p}_k \leqslant w_k\mathbf{1}_h \text{ each } (A_h, B_h, \mathbf{1}_h) \in H$$

where H is the set of all the existing systems of production and $(\mathbf{p}_k, w_k)$ are determined by the following equations:

$$[B_k - (1 + r)A_k]\mathbf{p}_k = w_k\mathbf{1}_k$$
$$\mathbf{q}^T\mathbf{p}_k = 1$$

Appendix to Section III

Definition 3.1 is not able to deal with some particular cases which can arise when joint production is involved. In another paper (Salvadori, 1982), where the theory of choice of technique within the Sraffa framework was provided, it was preferred to use another, less restrictive definition. It will be justified in this Appendix.

Two systems of production, whose operated methods neither pay extra profits nor require extra costs at rate of profit r and at prices of both systems, are said to be compatible at rate of profit r. This definition generalizes the concept of compatible systems commonly used (i.e., systems whose price-vectors are equal). Systems which have the same price-vectors are compatible, of course, but it is possible to find other cases, even among single product systems (consider two single product systems whose prices are all equal except for the price of a non-basic commodity which is not produced in both systems). This more general definition is particularly useful in studying joint production, since it is obviously possible that some process is not operated. Note, however, that if a system operates all its methods, then it is compatible with another system if and only if their price-vectors are equal. Finally:

Definition 3.2. A cost-minimizing system at rate of profit r is a system of production $(A, B, \mathbf{l})$ whose prices at rate of profit r are such that either no method pays extra profits or, if method $(\mathbf{a}, \mathbf{b}, l)$ does, all systems consisting of method $(\mathbf{a}, \mathbf{b}, l)$ and $n-1$ methods of system $(A, B, \mathbf{l})$ are compatible with it.

IV On Positive Prices

Neither in defining systems of production, nor in defining cost-minimizing systems, were prices assumed to be positive. This fact deserves further consideration.

Introducing joint production, Sraffa states:

> "only those methods of production are practicable which, in the conditions actually prevailing (i.e., at the given wage or at the given rate of profit) do not involve other than positive prices." (Sraffa, 1960, §50, p. 44.)

Therefore a reader may understand that some methods make up a system only if they generate a positive set of prices. I think, however, that this interpretation is false. In fact, after he has remarked that in the case of joint production some price may be negative at some feasible level of the wage (Sraffa, 1960, §69, p. 59), he asserts:

> "This conclusion is not in itself very startling. All that it implies is that, although in actual fact all prices were positive, a change in the wage might create a situation the logic of which required some of the prices to turn negative: and this being unacceptable, those among the methods of production that gave rise to such a result would be discarded to make room for others which in the new situation were consistent with positive prices." (Sraffa, 1960, §70, p. 59.)

If we consider the above quotations together, we see that they refer either to "*actual* fact" or to "conditions *actually* prevailing". So a system of production can involve negative prices but in the conditions actually prevailing they will be all positive. It can also be understood, from the second quotation, that the force pushing the economy to positive prices in actual fact is competition, i.e., choice of techniques. Sraffa, however, does not state that cost-minimizing systems have positive prices *by definition*: they are *expected* to have positive prices. Therefore, within the Sraffa framework, positiveness of the prices of cost-minimizing systems is a matter of proof, not one of definition. Sraffa, nevertheless, has never tried to prove his opinion. Section VI of this paper is devoted to investigating this opinion of Sraffa; in this section we have only been concerned with justifying the definitions previously given.

V A Comment on Section 96 of *Production of Commodities by Means of Commodities*[3]

The only problem of technique choice dealt with by Sraffa with respect to joint production is that of the choice among $k + 1$ systems made up by $k + 1$ processes and involving the production of k commodities. Sraffa asserts that if, in each system of production, no commodity price can fall faster than the wage rate as the profit rate rises, then the cost-minimizing system coincides, as in the single production case, with that system which allows the payment of a higher wage rate for each level of the profit rate and *vice versa* (see Sraffa, 1960, §96, pp. 86-87). The argument provided by Sraffa is not really convincing and the proposition is, in fact, false. This will be shown here by an example. In Section VIII the circumstances which permit the cost-minimizing system to differ from the wage-maximizing one will be further analysed.

Example 5.1. Let us consider a two commodity-three process economy whose input-output conditions are defined by Table 1. Requirements for use are defined by the following conditions: (a) the economy is stationary, i.e., net

Table 1

	Inputs				Outputs	
	Commodity 1	Commodity 2	Labour		Commodity 1	Commodity 2
Process (1)	1	1	2	→	2	4
Process (2)	—	3	1	→	3	—
Process (3)	—	1	2	→	2	—

[3]The material in this section was first presented in Italian (see Salvadori, 1979).

investment equals zero; (b) workers consume only commodity 2; and (c) capitalists consume only commodity 1.

It is easily shown that if $(\sqrt{33} - 3)/4 \leqslant r \leqslant (\sqrt{17} - 1)/2$, the processes (1) and (2) make up a system (system I); and if $(\sqrt{41} - 5)/2 \leqslant r \leqslant 2\sqrt{2} - 1$, the processes (1) and (3) make up another system (system II). By calculation, it is found that in each system no price falls faster than the wage rate. Sraffa's argument turns out to be false, since if $(\sqrt{41} - 5)/2 < r < 1$, then the cost-minimizing system exists but it does not coincide with the system which can pay the greater real wage rate.

In 1978, Sraffa himself suggested to me that the previous point could be clarified by the following argument: there exists a cement-plant producing only cement and there exists a steel-mill producing both cement and steel. Suppose now that a new, cheaper process producing only cement is introduced. Then the price of the cement in terms of the wage must fall. This fact reduces *both* costs and revenues of the steel-mill. Therefore the price of steel in terms of the wage may either rise or fall, in accordance with the rate of profit and the proportion in which cement enters into the inputs and into the outputs of the steel-mill. On the contrary, if the steel-mill produced only steel, *only* its costs would be reduced and, therefore, the price of steel in terms of wage would definitely be reduced.

VI On Positive Prices Once Again

As we have seen in Section IV, Sraffa is not startled by the possibility of negative prices within a joint product system. In fact, he argues that the choice of technique should push the economy to adopt a system of production consistent with positive prices. Sraffa's opinion was never proved and is, in fact, false. This will be shown here by an example. Further material will be provided in Section VIII.

Example 6.1. Let us consider a two commodity-three process economy whose input output conditions are defined by Table 2. Requirements for use are defined by the economy's being stationary and commodities' being consumed in the proportion of three units of commodity 1 to five units of commodity 2.

Table 2

	Inputs				Outputs	
	Commodity 1	Commodity 2	Labour		Commodity 1	Commodity 2
Process (1)	—	1	1	→	1	1
Process (2)	5	—	1	→	5	5
Process (3)	2	—	1	→	4	4

It is easily shown that the two existing systems are system I, made up by methods 1 and 2, and system II, made up by methods 1 and 3. Then, by simple calculation, it is found that if $0 \leqslant r < 2$, the cost-minimizing system exists and is system II, which, however, allows a negative price for $0 \leqslant r < \frac{1}{2}$, the prices of the other system being all positive.

Thus Sraffa's opinion, reviewed in Section IV, is false. And negative prices being unacceptable, we have to change something in the formalization of joint production proposed by Sraffa. Note that just to assume positiveness of prices cannot be a solution, since this increases the number of cases in which a cost-minimizing system does not exist. A solution to the problem which has been brought out in this section will be suggested in Section IX.

VII On The Existence of Cost-Minimizing Systems[4]

The theory of production prices provided by Sraffa (1960) is generally interpreted (see Garegnani, 1976, 1983) as the theory of the prices of the long-period position, studied by the Classical Economists and the first Neoclassical Economists, i.e., the prices which must prevail in the long run because of the persistent forces of competition. This interpretation, therefore, must entail that if a system of production exists, then there exists a cost-minimizing system. But this statement cannot be proved within the Sraffa framework if joint production is involved. This fact will be clarified here by an example and further material will be provided in Section VIII. In another paper (Salvadori, 1982) a sufficient condition for the existence of cost-minimizing systems within the Sraffa framework is provided (see also Bidard, 1984).

Example 7.1. Let us consider a two commodity-three process economy whose input-output conditions are defined by Table 3. Requirements for use are defined by the economy's being stationary and commodities' being consumed in proportions of one to one.

Table 3

	Inputs				Outputs	
	Commodity 1	Commodity 2	Labour		Commodity 1	Commodity 2
Process (1)	2	—	1	→	5	1
Process (2)	—	1	1	→	1	3
Process (3)	1	—	1	→	1	3

[4]The example provided in this section was first presented in Italian (see Salvadori, 1979, Appendix B). Other examples, but with no reference to requirements for use, have been supplied by Bidard (1984).

It is easily seen that the two existing systems are system I, made up by methods 1 and 2, and system II, made up by methods 1 and 3. Then, by simple calculation, it is found that if $0 \leqslant r < 1$, the cost-minimizing system exists and is system II. But if $1 < r < 9/5$, no system is cost-minimizing: process 3 is cheaper than process 2, at prices of system I, whereas process 2 is cheaper than process 3, at prices of system II.

The fact that a cost-minimizing system does not exist, even though a system of production exists, is remarkable. It implies that if Sraffa's formalization of joint production is accepted, then a long-period position does not necessarily exist. Therefore the following alternative holds: either we abandon the long-period tradition, or we refuse Sraffa's formalization of joint production, or both. Section IX suggests a way to save the long-period tradition by rejecting Sraffa's formalization of joint production and introducing a new formalization which will be recognized to be very similar to von Neumann's.

VIII Single and Joint Production: Some Differences

For the sake of completeness, in this section we will compare the single and joint production cases with respect to the problem of the choice of techniques, with a given rate of profit, in a two commodity-three process economy. A general treatment of the choice of techniques problem with a given profit rate was provided by Salvadori (1982) and Bidard (1984). Some suggestions towards solving the problem of choice of techniques with a given wage rate can be found in Salvadori (1981).

In a two commodity economy, an operating process determines the following constraints between the wage rate, w, and the prices of commodities, p_1 and p_2:

$$ap_1 + bp_2 + wl = cp_1 + dp_2$$

where a and b are the inputs multiplied by the interest factor $1 + r$, r being the rate of profit, c and d are the outputs of commodities 1 and 2 respectively, and l is the input of labour.

Assume, without loss of generality, that commodity 2 is consumed by workers: if this is not the case, interchange the names of commodities. Then we can normalize prices by setting

$$\lambda p_1 + p_2 = 1,$$

where λ is the amount of commodity 1 workers consume for each unit of commodity 2 consumed. This numéraire has the advantage that the wage rate is expressed in real terms.

Thus, a process can be indicated on the plane (p_1, w) by the straight line defined by the following equation

$$lw = (d - b) + \Delta p_1 \qquad \text{......(8.1)}$$

where

$$\Delta = \det \begin{bmatrix} c-a & d-b \\ \lambda & 1 \end{bmatrix} \qquad \text{......(8.2)}$$

The following statements are trivially proved:

(i) if $l = \Delta = d - b = 0$, then equation (8.1) is satisfied by each point of the whole plane (p_1, w);

(ii) if $l = \Delta = 0$, $d - b \neq 0$, then equation (8.1) is satisfied by no point;

(iii) if $l = 0$, $\Delta \neq 0$, then curve (8.1) is a vertical straight line;

(iv) if $\Delta = 0$, $l \neq 0$, then curve (8.1) is a horizontal straight line;

(v) if $l > 0$, and $\Delta > 0$ $(\Delta < 0)$, curve (8.1) is an increasing (a decreasing) straight line.

It is also immediately proved that a process producing only commodity 1 $(c > a, d = 0)$ and utilizing labour defines on plane (p_1, w) an increasing straight line which cuts the vertical axis at a non-positive value of w, whereas a process producing only commodity 2 $(c = 0$ and $d > b)$ defines, on the same plane, a decreasing curve which cuts the vertical axis at a non-negative value of w.

In Fig. 1 there are two single product processes forming a system of

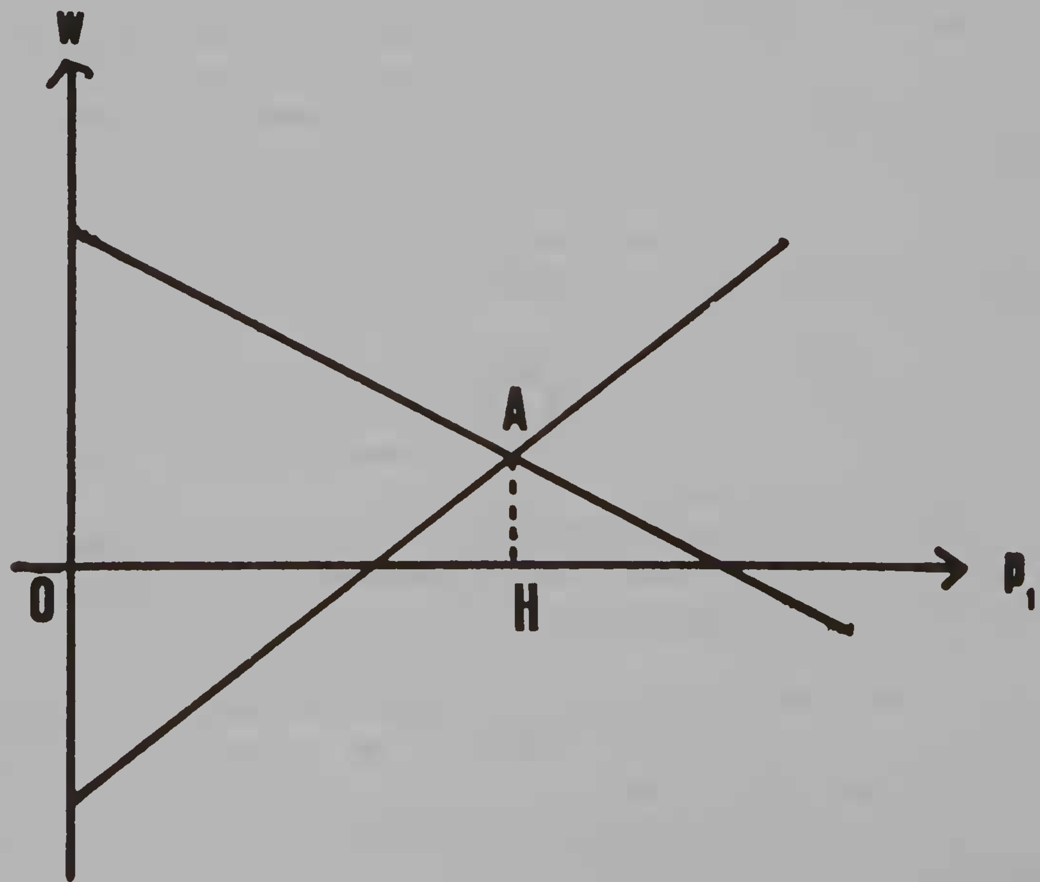


Fig. 1

production. The wage and prices are defined by point A ($w = \overline{HA}$, $p_1 = \overline{OH}$, $p_2 = 1 - \lambda\overline{OH}$). In Fig. 2 another process producing only commodity 1 is introduced into the analysis. If prices are defined by point A, capitalists who operate the new process can obtain an extra profit equal to $\overline{BA}$. When the obsolete process is suppressed the wage rate and the prices are determined by point C ($w = \overline{KC}$, $p_1 = \overline{OK}$, $p_2 = 1 - \lambda\overline{OK}$). Note that since C is on the decreasing straight line DE (on the increasing straight line IL) $\overline{KC}$ must be greater than $\overline{HA}$ (less than $\overline{HB}$). A similar argument also applies if the new process produces only commodity 2.

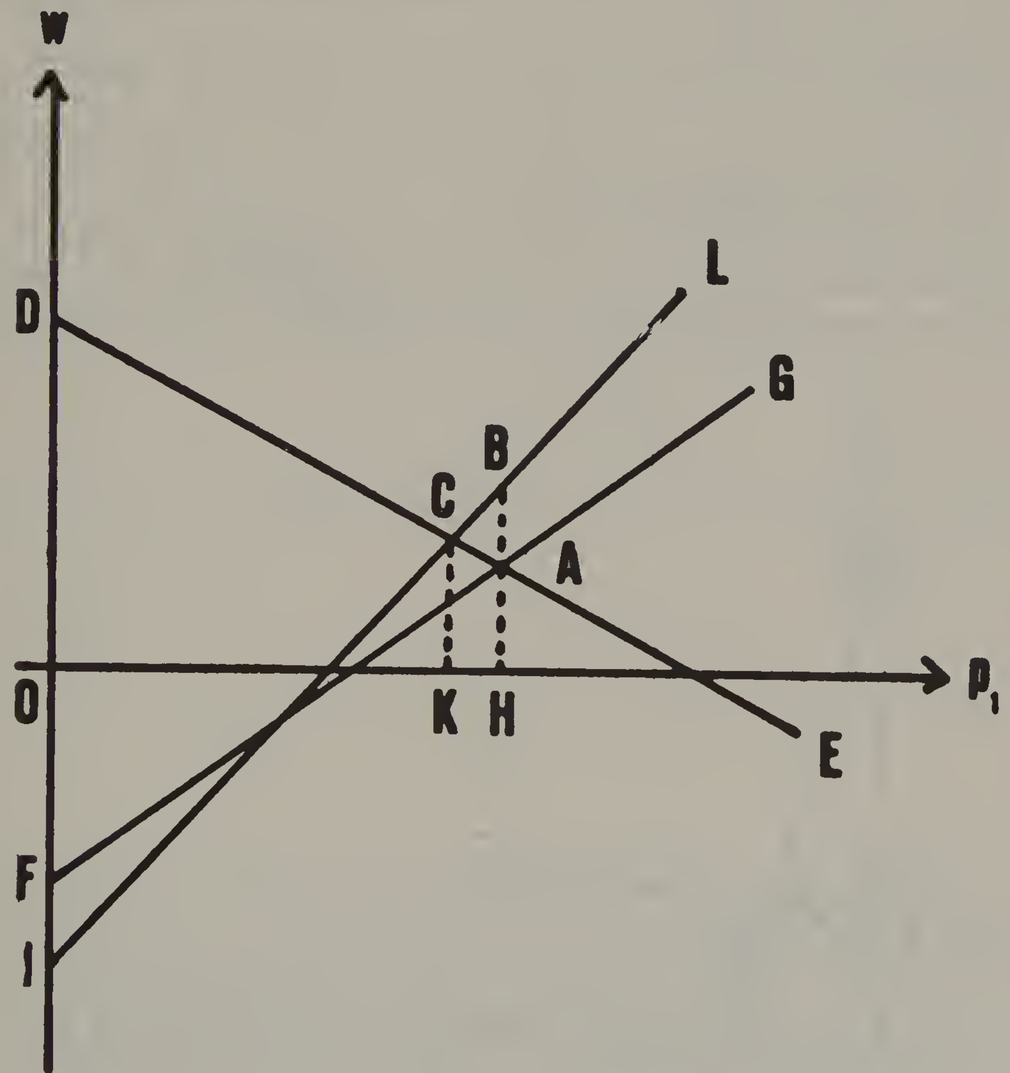


Fig. 2

Note that if both prices are positive in the non-cost-minimizing system, they must be positive in the cost-minimizing one. In fact, if the alternative processes produce commodity 1, then the price of commodity 1 is lower in the cost-minimizing system than in the other, but still positive, the price of commodity 2 being higher ($\lambda > 0$) or equal ($\lambda = 0$). If the alternative processes produce commodity 2 and $\lambda > 0$, then interchange names of commodities and the same argument applies. Finally, if the alternative processes produce commodity 2 and $\lambda = 0$, then the price of commodity 2 is equal to unity in both systems and the price of commodity 1 is higher in the cost-minimizing system than in the other.

The results obtained by analysing Fig. 2 can be summarized in the following way. In a two commodity-three process economy, the usual results are obtained if all the following three statements hold:

(a) non-alternative processes have, on plane (p_1, w), straight lines with opposite sign slopes;

(b) alternative processes have, on plane (p_1, w), straight lines with the same sign slopes;

(c) increasing (decreasing) straight lines of type (8.1) cut the vertical axis at a negative (positive) value of w.

In the general joint production case statements (a), (b), and (c) do not need to hold (note, however, that if (a) holds, (b) holds also). If (b) does not hold, then a cost-minimizing system does not need to exist (see Fig. 3, where

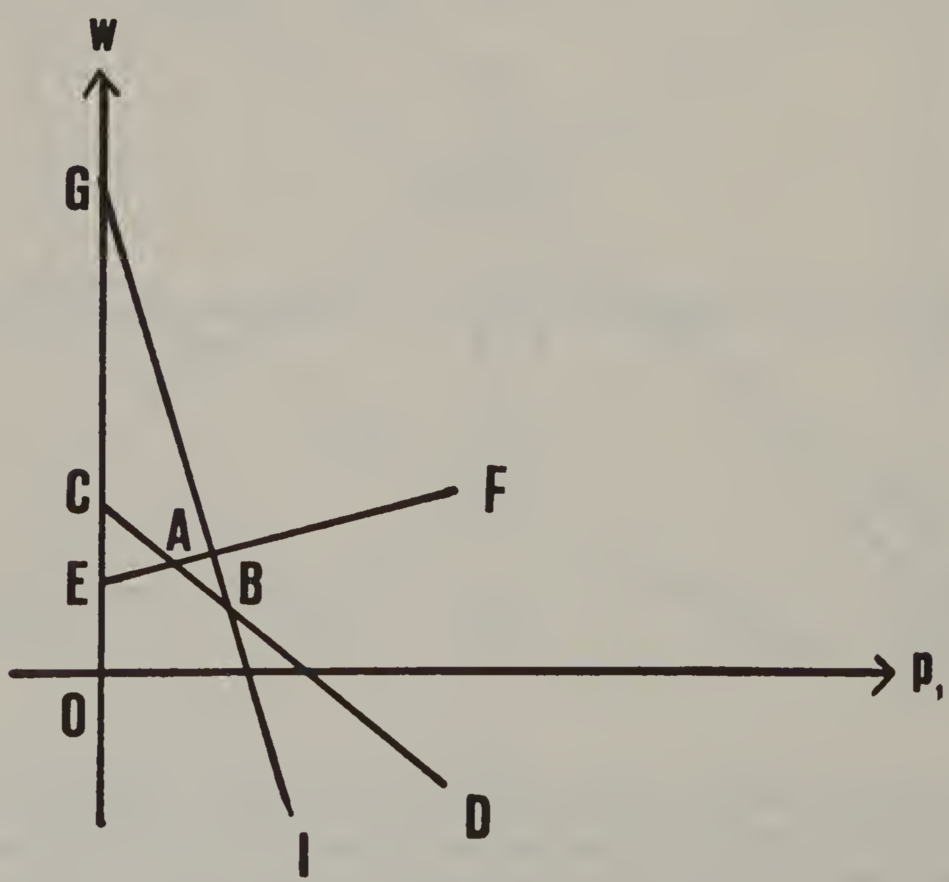


Fig. 3

curves CD, EF, GI relate to processes (1), (2), (3) of Example 7.1, $r = 3/2$). It is also possible that there exist two non-compatible cost-minimizing systems (see Fig. 4). If (b) holds and (a) does not hold, then there exists a cost-minimizing system, but the wage is lower in the cost-minimizing system

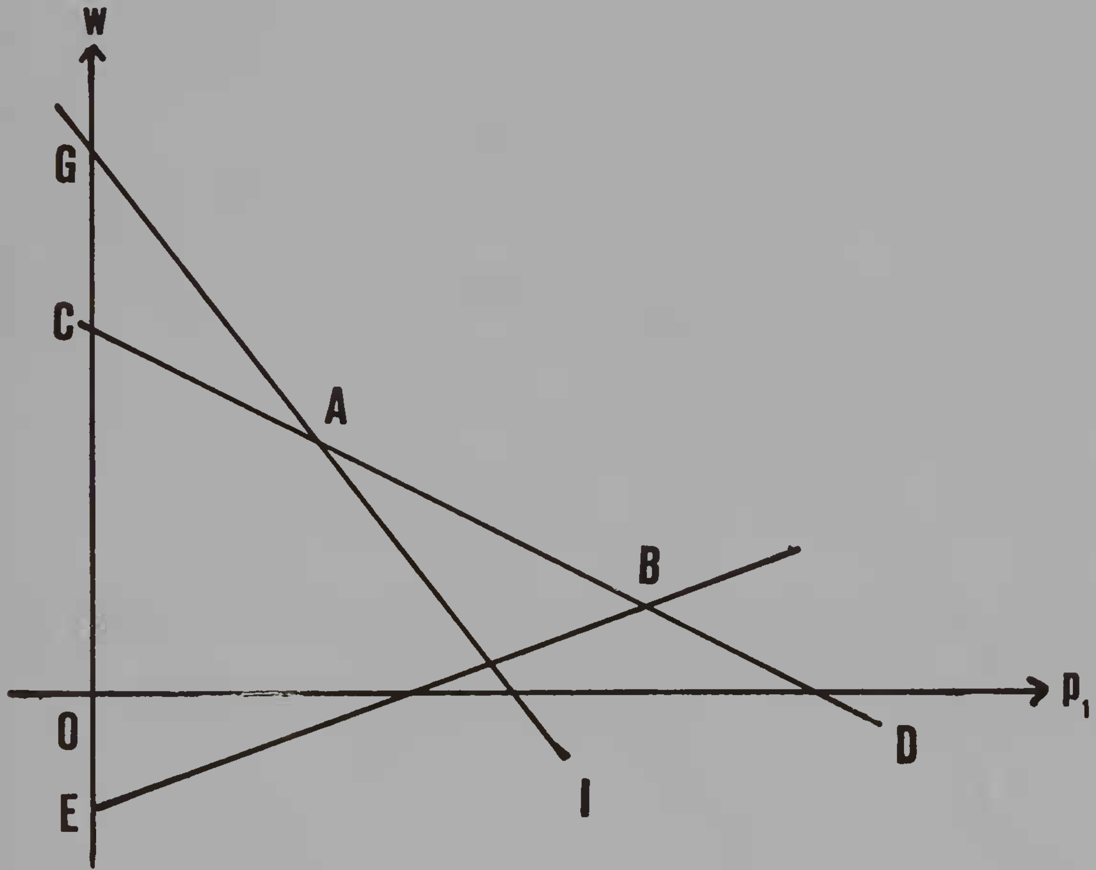


Fig. 4

than in the other (see Fig. 5, where curves DE, FG, IL relate to process (1), (2), (3) of Example 5.1, $r = \frac{1}{2}$). If (b) holds and (c) does not hold, there exists a cost-minimizing system, but some commodity may have a negative price in the cost-minimizing system even though all prices are positive in the other system (see Fig. 6, where curves CD, EF, GI relate to processes (1), (2), (3) of Example 6.1, $r = \frac{1}{4}$).

Let us conclude this section by remarking that if two processes determine, on plane (p_1, w), straight lines with the opposite (same) sign slope, then it is (not) possible to combine them to obtain an economy growing at a uniform growth rate equal to the profit rate, where commodities 1 and 2 are consumed

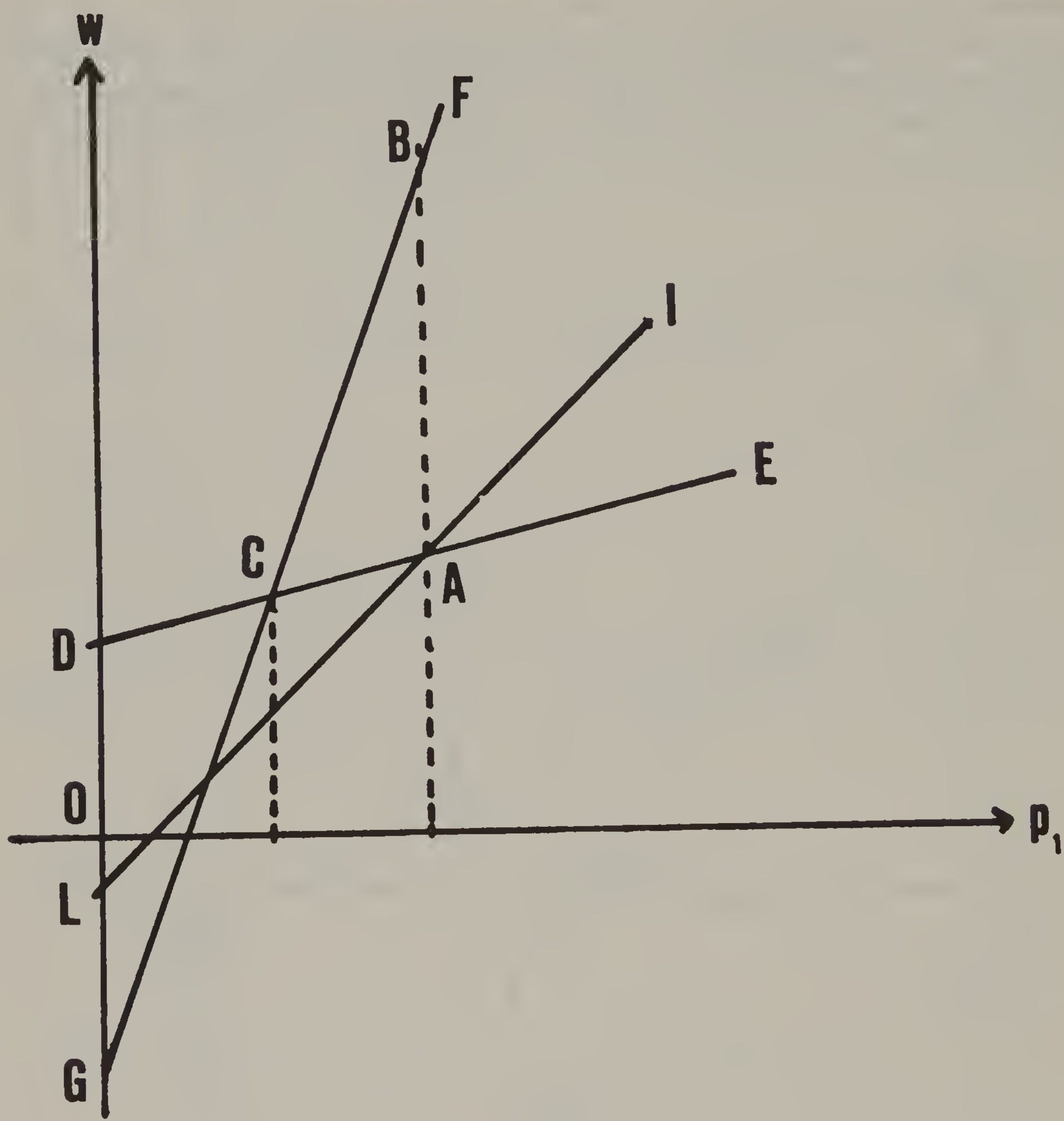


Fig. 5

in proportions λ to 1.[5] Therefore, (b) holds if and only if either both alternative systems would or both would not still be systems if the requirements for use were those of an economy growing at a uniform growth rate equal to the profit rate where commodities 1 and 2 are consumed in proportion of

[5]In order to see this, it is sufficient to consider equations (8.1) and (8.2) keeping in mind that

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} . \det \begin{bmatrix} \varepsilon & \varphi \\ \gamma & \delta \end{bmatrix} < 0$$

if and only if the points (α, β) and (ε, φ) are contained in two open half-planes produced by the straight line which contains the origin and the point (γ, δ).

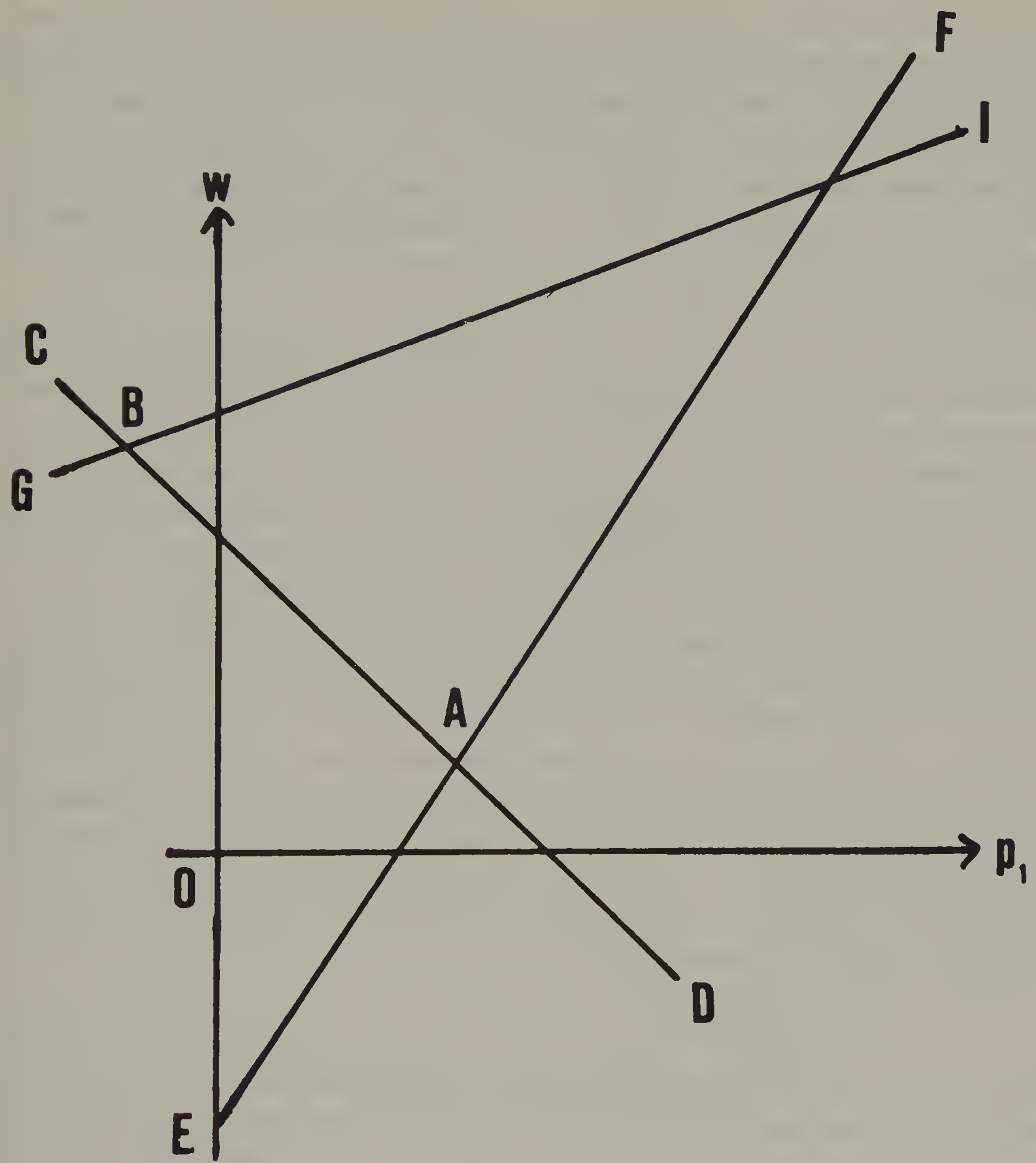


Fig. 6

λ to 1.[6] In the former case (a) holds too; in the latter (a) does not hold.

IX A Critique and a Reformulation

Section VI has shown that Sraffa's opinion (1960, §70) notwithstanding, cost-minimizing systems may have negative prices, even if some other

[6]It is possible to prove that in a k commodity, $(k + 1)$ process economy there exists a cost-minimizing system if and only if there exists a bundle of commodities such that each system of production can be arranged in such a way that the economy could consume that bundle of commodities and grow at a uniform growth rate equal to the profit rate; moreover, if there exist more than one cost-minimizing system they are compatible with each other (see Salvadori, 1982, §4).

non-cost-minimizing system does not. Section VII has shown that cost-minimizing systems do not need to exist even if some system exists. And Section VIII has shown when these difficulties may arise. It is obvious that the method of long-period positions (see Garegnani, 1976) and a theory which permits such results cannot cohere. Therefore we must reject either the former or the latter or both. This section will suggest a way out by introducing new definitions of "system of production" and "cost-minimizing system".

Definition 2.1 states that a system of production *à la* Sraffa is a set of processes which has two properties: the former (exact satisfaction of requirements for use) is required by realism, the latter (equality between the number of processes and the number of commodities involved) is required by the form of analysis: otherwise prices cannot be determined (*c.f.* Sraffa, 1960, §50). Therefore we cannot drop the former, but we may drop the latter if we substitute for it another condition which is able to determine prices.

If the number of processes is allowed to be less than the number of commodities involved, then requirements for use can be satisfied *exactly* only by a fluke. Therefore requirements for use must be allowed to be satisfied in excess (so that the former condition is also partially weakened). If the price of a commodity produced in excess with respect to requirements for use were positive (negative), the rate of profit actually obtained by capitalists producing that commodity would be less (more) than the current rate of profit. Let $(\mathbf{a}, \mathbf{b}, l)$ be an operated method of production; then the following constraints must hold

$$r = \frac{\mathbf{b}^T\mathbf{p} - \mathbf{a}^T\mathbf{p} - wl}{\mathbf{a}^T\mathbf{p}}$$

where r, w and $\mathbf{p}$ are the profit rate, the wage rate and the price vector respectively (however they are determined). On the contrary, the rate of profit actually obtained by capitalists operating method $(\mathbf{a}, \mathbf{b}, l)$ is

$$\hat{r} = \frac{\hat{\mathbf{b}}^T\mathbf{p} - \mathbf{a}^T\mathbf{p} - wl}{\mathbf{a}^T\mathbf{p}}$$

where $\hat{\mathbf{b}}$ ($\hat{\mathbf{b}} \leqslant \mathbf{b}$) is the vector of outputs actually sold. It is obvious that if $\hat{\mathbf{b}} \neq \mathbf{b}$, and if prices of commodities produced in excess were not equal to zero, then

$$r = \hat{r}$$

would hold only by a fluke. Thus we have to assume that the price of a commodity produced in excess is equal to zero.[7] Hence we can give the

[7]I am indebted to Heinz Kurz for very useful remarks on this point.

following definition:

Definition 9.1.[8] A system of production is a non-empty set of processes which can be operated in such a way that requirements for use are satisfied and prices are determined for some value of the rate of profit, following the rule that if the produced quantity of a commodity exceeds the requirements for use, its price is equal to zero.

It may be remarked that once one of the two distribution variables is exogenously given, prices are not determined, with the Definition 9.1, by the conditions of production only. Conditions of production and requirements for use concur in determining prices: systems made up by the same processes but with different requirements for use do not need to have the same prices.

Let us turn now to the definition of "a cost-minimizing system". Definition 9.1 does not allow non-operated processes, so we do not have to mention them in the definition of cost-minimizing systems. It could be as follows: a cost-minimizing system at rate of profit r is a system whose prices at rate of profit r are such that no process can pay extra profits. But negative prices could not thereby be avoided (see Example 6.1). Negative prices being unacceptable, a cost-minimizing system will be assumed to have non-negative prices.

Definition 9.2. A system is cost-minimizing at rate of profit r if its prices at rate of profit r are non-negative and such that no process can pay extra profits at those prices.

In order to put the matter more formally, let us assume that there exist m methods of production which involve the production of n commodities. Let $A(B)$ be the $m \times n$ matrix of inputs (outputs), whose element a_{ij} (b_{ij}) is the input (output) of commodity j in process i $(j = 1, 2, \ldots, n;$ $i = 1, 2, \ldots, m)$. Let $\mathbf{l}$ be the m-vector of labour, whose element l_i is the input of labour in process i. Then the two previous definitions can be given in the following way.

Definition 9.3. The set of processes s, $s \subset \{1, 2, \ldots, m\}$, is a system of production at rate of profit r, with respect to requirements for use $\mathbf{d}$, if there exists an m-vector $\mathbf{x}$, an n-vector $\mathbf{p}$ and a scalar w such that:

$$x_i = 0 \qquad \text{if } i \notin s \qquad \ldots\ldots(9.1)$$

$$\mathbf{x}^T (B - A) \geqslant \mathbf{d}^T \qquad \ldots\ldots(9.2)$$

$$\text{if } \mathbf{x}^T (B - A)\mathbf{e}_i > \mathbf{d}^T\mathbf{e}_i \text{ then } \mathbf{e}_i^T\mathbf{p} = 0 \qquad \ldots\ldots(9.3)$$

$$\mathbf{x}^M [B - (1 + r)A]\mathbf{p} = w\mathbf{x}^M\mathbf{l} \qquad \ldots\ldots(9.4)$$

[8]If the requirements for use are those of an economy growing at a uniform growth rate equal to the profit rate and consuming the same bundle of commodities whatever prices are, Definition 9.1 coincides with the definition of "quadratic system or truncation" utilized by Schefold (1978b, 1980).

$$\mathbf{x} \geqslant 0, \qquad \text{......(9.5)}$$

where $\mathbf{x}^M$ is a diagonal matrix having the elements of vector $\mathbf{x}$ on the principal diagonal (i.e., $x_{ii}^M = x_i$) and $\mathbf{e}_i$ is the ith unit vector, i.e., a vector whose elements are all equal to zero but for the ith which is equal to 1.

Definition 9.4. The system of production s is cost-minimizing at rate of profit $\bar{r}$ if vectors $\mathbf{x}$ and $\mathbf{p}$ and scalar w, satisfying constraints (9.1) – (9.5) for $r = \bar{r}$, also satisfy the following inequalities:

$$[B - (1 + \bar{r})A]\mathbf{p} \leqslant w\mathbf{l} \qquad \text{......(9.6)}$$

$$\mathbf{p} \geqslant 0 \qquad \text{......(9.7)}$$

Now we are able to recognize that there are two ways to determine a cost-minimizing system. One is the following: First, find all systems of production, i.e., all subsets $s_1, s_2, \ldots, s_z (z \leqslant \sum_{k=1}^{m} \binom{m}{k})$ of the set $\{1, 2, \ldots, m\}$ such that vectors $\mathbf{x}$ and $\mathbf{p}$ and scalar w satisfying constraints (9.1) – (9.5) exist; second, find if there exists a system of production whose prices satisfy inequalities (9.6) – (9.7).

A more direct way is the following: find if there exist vectors $\mathbf{x}$ and $\mathbf{p}$ and scalar w satisfying the constraints (9.2) – (9.7); if $\mathbf{x}^\circ$ and $\mathbf{p}^\circ$ are such a pair of vectors, then the set of processes

$$s = \{i / i \in \{1, 2, \ldots, m\}, x_i^0 \neq 0\} \qquad \text{......(9.8)}$$

is a cost-minimizing system; if there is no pair of vectors satisfying constraints (9.2) – (9.7), no cost-minimizing system exists.

Thus it is possible to prove the following theorem (the proof is trivial).

Theorem 9.1. For m given methods of production defined by output-input matrices A, B and by labour-vector $\mathbf{1}$, and for requirements for use $\mathbf{d}$, there exists a cost-minimizing system at rate of profit r if and only if there exist two vectors $\mathbf{x}$ and $\mathbf{p}$ and a scalar w such that:

$$\mathbf{x}^T (B - A) \geqslant \mathbf{d}^T \qquad \text{......(9.9)}$$

$$\mathbf{x}^T (B - A)\mathbf{p} = \mathbf{d}^T\mathbf{p} \qquad \text{......(9.10)}$$

$$[B - (1 + r)A]\mathbf{p} \leqslant w\mathbf{l} \qquad \text{......(9.11)}$$

$$\mathbf{x}^T [B - (1 + r)A]\mathbf{p} = w\mathbf{x}^T\mathbf{1} \qquad \text{......(9.12)}$$

$$\mathbf{x} \geqslant 0 \qquad \text{......(9.13)}$$

$$\mathbf{p} \geqslant 0 \qquad \text{......(9.14)}$$

If $(\mathbf{p}^\circ, \mathbf{x}^\circ, w^\circ)$ is a solution of inequality system (9.9) – (9.14), the set of methods (9.8) is a cost-minimizing system.

Theorem 9.1 is remarkable since the inequality system (9.9) – (9.14) can be interpreted as a generalized von Neumann model. To show this it is sufficient to remark that if a uniform growth rate g is assumed to exist, then requirements for use can be written in the following way:

$$\mathbf{d}^T = g\mathbf{x}^T A + \mathbf{c}^T \qquad \ldots\ldots(9.16)$$

where $\mathbf{c}$ is the consumption vector. By putting $\mathbf{d}$ as defined in equation (9.15) in system (9.9) – (9.14) we obtain the generalized von Neumann model introduced by Morishima (1960), and studied by Morishima (1964, 1969), and others (see, for instance, Burmeister, 1974; Haga-Otsuki, 1965; Salvadori, 1980).

X Concluding Remarks

If switching in methods of production is involved, Sraffa's theory of joint production encounters some difficulties. In this paper, this has been shown and a solution suggested. Such a solution works for particular hypotheses on requirements for use (see the results quoted at the end of the previous section), but does not work for all requirements for use. By examples[9] it is possible to show that the formal properties of the function of requirements for use do matter in proving the existence of cost-minimizing systems. So one might think that, once again, some assumption must be changed. But I do not think this is the case because requirements for use play a role in determining prices: what we now need could be, in fact, a plausible theory of requirements for use. At the same time, finding further conditions sufficient for the existence of cost-minimizing systems would be useful too.

References

Baldone, S. (1980). "Fixed Capital in Sraffa's Theoretical Scheme" in L. L. Pasinetti (ed.), *Essays on the Theory of Joint Production*, London, Macmillan, pp. 88-137.

Bidard, C. (1984). "Choix Techniques en Production Jointe" in C. Bidard (ed.) *La Production Jointe, Nouveaux Débats*, Paris, Economica, pp. 186-207.

Burmeister, E. (1974). "Synthesizing Neo-Austrian and Alternative Approaches to Capital Theory: a Survey", *Journal of Economic Literature*, Vol. 12, No. 2, pp. 413-456.

Garegnani, P. (1973). "Nota Matematica", published as appendix to the Italian edition of "Beni capitali eterogenei, la funzione della produzione e la teoria della distribuzione" in P. Sylos Labini (ed.), *Prezzi Relativi e Distribuzione del Reddito*, Turin, Boringhieri. (This mathematical note was mentioned but not published in the original English edition of the paper published in the *Review of Economic Studies*, 1970, Vol. 37, No. 109, pp. 407-436.)

[9]Such an example was provided by Salvadori (1980, Appendix).

Garegnani, P. (1976). "On a Change in the Notion of Equilibrium in Recent Work on Value and Distribution: a Comment on Samuelson" in M. Brown, K. Sato, P. Zarembka (eds.), *Essays in Modern Capital Theory*, Amsterdam, North-Holland, pp. 25-45.

———(1984). "Value and Distribution in the Classical Economists and Marx", *Oxford Economic Papers*, Vol. 36, No. 2, pp. 291-325.

Haga, H. and Otsuki, M. (1965). "On a Generalized von Neumann Model", *International Economic Review*, Vol. 6, No. 11, pp. 115-123.

Kurz, H. D. (1980). "Smithian Themes in Piero Sraffa's Theory", *Journal of Post-Keynesian Economics*, Vol. 3, No. 2, pp. 271-80.

Levhari, D. (1965). "A Non Substitution Theorem and Switching of Techniques", *Quarterly Journal of Economics*, Vol. 79, No. 1, pp. 98-105.

Lippi, M. (1979). *I prezzi di produzione. Un saggio sulla teoria di Sraffa*, Bologna, Il Mulino.

Łos, J. and Łos, M. W. (1976). "Reswitching of Techniques and Equilibria of Extended von Neumann Models" in J. Łos, M. W. Łos and A. Wieczorek (eds.), *Warsaw Fall Seminars in Mathematical Economics 1975*, Berlin, Springer-Verlag, pp. 97-118.

Morishima, M. (1960). "Economic Expansion and the Interest Rate in Generalized von Neumann Models", *Econometrica*, Vol. 28, No. 2, pp. 352-363.

——— (1964). *Equilibrium, Stability and Growth: A Multisectoral Analysis*, Oxford, Clarendon Press.

——— (1969). *Theory of Economic Growth*, Oxford, Clarendon Press.

Roncaglia, A. (1978). *Sraffa and the Theory of Prices*, New York, John Wiley and Sons.

Salvadori, N. (1979). "Mutamento dei metodi di produzione e produzione congiunta. Un commento al §96 di *Produzione di merci a mezzo di merci*", *Studi Economici*, Vol. 34, No. 7, pp. 79-94.

——— (1980). "On a Generalized von Neumann Model", *Metroeconomica*, Vol. 32, No. 1, pp. 51-62.

——— (1981). "Falling Rate of Profit with a Constant Real Wage. An Example", *Cambridge Journal of Economics*, Vol. 5, No. 1, pp. 59-66.

——— (1982). "Existence of Cost-Minimizing Systems within the Sraffa Framework", *Zeitschrift für Nationalökonomie*, Vol. 42, No. 3, pp. 281-298.

Schefold, B. (1977). "Fixed Capital as a Joint Product", *Jahrbucher für Nationalökonomie und Statistik*, Vol. 192, No. 5, pp. 415-439.

——— (1978a). "Multiple Product Techniques with Properties of Single Product Systems", *Zeitschrift für Nationalökonomie*, Vol. 38, Nos. 1-2, pp. 29-53.

——— (1978b). "On Counting Equations", *Zeitschrift für Nationalökonomie*, Vol. 38, Nos. 3-4, pp. 253-285.

——— (1980). "Von Neumann and Sraffa: Mathematical Equivalence and Conceptual Difference", *Economic Journal*, Vol. 90, No. 357, pp. 140-156.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge, Cambridge University Press.

Steedman, I. (1980). "Returns to Scale and the Switch in Methods of Production", *Studi Economici*, Vol. 34, No. 10, pp. 5-10.

Varri, P. (1980). "Prices, Rate of Profit and Life of Machines in Sraffa's Fixed Capital Model" in L. L. Pasinetti (ed.), *Essays on the Theory of Joint Production*, London, Macmillan, pp. 55-87.

METROECONOMICA

Vol. XXVIII Gennaio-Dicembre 1976 Fasc. I-II-III

REDUCTION TO DATED QUANTITIES OF LABOUR, ROUNDABOUT PROCESSES, AND SWITCHES OF TECHNIQUE IN FIXED CAPITAL SYSTEMS

by Bertram Schefold

This paper examines a formal simplification of fixed capital systems of the Sraffa type [1] by means of an extension of the reduction to dated quantities of labour. The reduction will be used to discuss particular types of switches of techniques which arise only in the context of fixed capital systems.

1. Reduction

In the case of single product industries the reduction to dated quantities of labour consists in resolving the price of a product into the series of the past inputs of direct and indirect labour which have gone into it. Each labour input is multiplied by the wage rate w (measured in the standard chosen) and a power of the rate of profit r indicating the number of periods which have elapsed between the expenditure of that amount of labour and its embodiment in the product. Take e.g. prices $\hat{p} = p/w$ in terms of the wage rate in an indecomposable single product system with input matrix A and labour vector l. The mathematical series for dated inputs can then be derived as follows:

$$\begin{aligned}\hat{p}(r) = \frac{1}{w} p(r) &= l + (1+r) A\hat{p}(r) \\ &= l + (1+r) Al + (1+r)^2 A^2 \hat{p}(r) \\ &= l + (1+r) Al + \dots + (1+r)^n A^n \hat{p}(r),\end{aligned}$$

hence

$$\hat{p}(r) = l + (1+r) Al + (1+r)^2 A^2 l + \dots,$$

for it is well known that this series converges for $0 \leq r < R$, if A is productive, i.e. if dom $A = (1+R)^{-1} < 1$, where R is the maximum

(1) P. Sraffa [1960].

— 2 —

rate of profit. The term $A^n l$ denotes the vector of labour inputs expended n periods ago which enter indirectly a vector of unit outputs today. Mr. Sraffa has shown in chapter VI of his book that the erratic character of the movements of relative prices in response to changes in the rate of profit is intuitively best understood by considering the irregular pattern of the distribution of direct and indirect labour inputs over the past. By so doing he has proved at the same time that prices of basic products cannot be ultimately resolved in pure labour time by means of a finite number of steps, contrary to the assumptions of the original Austrian theory of capital.

Unfortunately, the same reduction does in general not converge in a general joint production system, where the unit output matrix I of the single product system has been replaced by a quadratic output matrix B, for the series

$$\begin{aligned}\hat{p} &= (B - (1+r)A)^{-1} l = (I - (1+r)B^{-1}A)^{-1} B^{-1} l \\ &= B^{-1} l + (1+r) B^{-1} A B^{-1} l + (1+r)^2 (B^{-1}A)^2 B^{-1} l + \dots\end{aligned}$$

which is the closest analogue to Mr. Sraffa's reduction may not even converge for $r = 0$. The series

$$\begin{aligned}\hat{p} &= (B - (1+r)A)^{-1} l = [(B - A)(I - r(B-A)^{-1} A]^{-1} l \\ &= (I - r(B-A)^{-1} A)^{-1} (B-A)^{-1} l \\ &= (B-A)^{-1} l + r(B-A)^{-1} A (B-A)^{-1} l + \\ &\qquad + r^2 [(B-A)^{-1} A]^2 (B-A)^{-1} l + \dots\end{aligned}$$

does always converge for sufficiently small r, but not necessarily in the full range of r where prices are positive and well defined. Moreover, some of its terms will usually be negative, since $(B - A)^{-1}$ is generally not positive for joint production systems. It is true that the single terms of the series possess an economic meaning [2] but I suggest that yet another series provides a better understanding of the specific features of *fixed capital systems*.

2. Fixed Capital and the 'Centre'

The input matrix A and the output matrix B of a fixed capital system have a special structure expressing the fact that each process uses a $t-1$ years old machine to produce one finished good (which may be a consumption good, a raw material, or a new — zero years old — machine) plus a t years old machine. Using a suitable notation we may say that finished good i ($i = 1, \dots, f$) with price p_i is thus in

---

[2] $(B - A)^{-1} l$ is the total of direct and indirect labour expended on one unit of each good. The subsequent expressions can be explained in terms of integrated industries, as L. Pasinetti [1973] has shown.

— 3 —

year t produced by means of finished goods (raw materials) $a^j_i(t)$, $j = 1, \dots, f$, labour $l_i(t)$ and a machine $m_i(t-1)$ with price $p_{m,i}(t-1)$ lasting a specified number of years T_i $(t = 1, \dots, T_i)$. The price-equation for year t is:

$$(1+r)\,(m_i(t-1)\,p_{m,i}(t-1) + a^1_i(t)\,p_1 + \dots$$
$$\dots + a^f_i(t)\,p_f) + wl_i(t) = b^i_i(t)\,p_i + m_i(t)\,p_{m,i}(t)\,.$$

With this definition we exclude 'superimposed joint production' where a process may produce more than one finished good (we have $b^j_i(t) = 0$, $i \neq j$). This is justified, if we want to separate the problem of fixed capital from that of joint production in general. We are also assuming that old machines are only used in the 'group' of processes producing the same finished good, and this implies that we do not consider transfers of old machines between groups. It is therefore not meaningful to calculate separate prices for various pieces of equipment which are used in the group; the expression $m_i(t)$ is rather to be considered as a symbol for the whole complex of durable equipment which has to be used in the production of good i in year t. This assumption may seem arbitrary. It is natural in some cases (there is no meaningful price of production for an old assembly line independently of the factory where it is located), but it is in fact arbitrary in others (e.g. in the case of old typewriters). We adopt the assumption, nevertheless, since it allows an economically relevant generalization of single product systems without leading into some of the complexities of joint production proper (³). It is a useful convention to assume that $m_i(0) = 0$, since the new machine is a complex of finished goods contained in $(a^1_i(1), \dots, a^f_i(1))$. Of course $m_i(T_i) = 0$, and, as a normalization without loss of generality, $m_i(1) = \dots = m_i(T_i - 1) = 1$. Total output of each commodity is also normalized to one, i.e.

$$\sum_{t=1}^{T_i} b^i_i(t) = 1\,; \qquad i = 1, \dots, f\,;$$

and, since the system is assumed to produce a surplus:

$$\sum_{i=1}^{f} \sum_{t=1}^{T_i} a^j_i(t) \leq 1\,; \qquad j = 1, \dots, f\,;$$

with the inequality holding for at least one j. The $a^j_i(t)$ will in general be different for different t, given i, j, which means (if $a^j_i(1)$ does not itself denote a 'machine') that the efficiency of the 'machine' in group i varies. Of course $a^j_i(t) = 0$, $t = 2, \dots, T_i$, if $a^j_i(1)$ denotes a machine which is turned into $m_i(1)$.

(³) See B. Schefold: Theorie der Kuppelproduktion, Basel 1971 (private print).

— 4 —

For an entrepreneur looking back at the end of year T_i when his machine has worn out physically, it must be true that the sum of his expenses for means of production including labour must be equal to the sum of his revenues, each multiplied with the power of $(1+r)$ which indicates the number of years since the expense was made or the revenue received, therefore

$$\sum_{t=1}^{T_i}\left(\sum_{j=1}^{f}\{b^j{}_i(t)-(1+r)\,a^j{}_i(t)\}\,p_j-wl_i(t)\right)(1+r)^{T_i-t}=0\,,\qquad i=1,\dots,f.$$

These equations are more formally also obtained, if we imagine the processes constituting group i running competitively side by side; and if we add up the processes after premultiplying each with the appropriate power of $(1+r)$. The resulting equations are called the 'integrated processes'. Old machines have disappeared from them. One can show that they determine the prices of finished goods uniquely and that these prices are positive between $r=0$ and a maximum rate of profit R, provided only that the system is basic and capable of producing a surplus. These prices of finished goods can then be used to examine the prices of old machines, given various efficiency patterns of the machines. Prices of old machines may be negative, but if the system is capable of producing a surplus, it will always be possible to truncate the lifetimes of the machines. After the truncations which yield the maximum real wage at each rate of profit have been made, prices of finished goods in terms of the wage-rate will rise continuously and monotonically with the rate of profit, and negative prices of old machines will have disappeared (4).

We shall now assume that optimal truncations have been made. Using the abbreviations

$$\hat{a}^j{}_i(r)=\frac{\sum_{t=1}^{T_i}(1+r)^{T_i-t}\,a^j{}_i(t)}{\sum_{t=1}^{T_i}(1+r)^{T_i-t}\,b^i{}_i(t)}$$

$$\hat{l}_i(r)=\frac{\sum_{t=1}^{T_i}(1+r)^{T_i-t}\,l_i(t)}{\sum_{t=1}^{T_i}(1+r)^{T_i-t}\,b^i{}_i(t)}$$

the equation of the i-th integrated process becomes

$$(1+r)\,\hat{a}_i(r)\,p+w\hat{l}_i(r)=p\,,\qquad \hat{a}_i=(\hat{a}^1{}_i,\dots,\hat{a}^f{}_i)\,,$$

(4) These propositions are proved in Schefold (1974) and some of them were proved earlier in Schefold (1971).

— 5 —

or in matrix form

$$(1 + r)\, \hat{A}\,(r)\, p + w\hat{l}\,(r) = p$$

where

$$\hat{A} = (\hat{a}^j{}_i(r))\,, \quad \hat{l}\,(r) = (\hat{l}_1\,(r)\,, \dots, \hat{l}_l\,(r)\,)'\,, \quad p = (p_1\,, \dots, p_l)'\,,$$

(the dash (′) denotes the transposed of a vector or matrix).

We have thus constructed sort of a single product system with variable coefficients. We call it the 'centre' and its coefficients the centre coefficients. $\hat{A}\,(r)$ is an indecomposable matrix for each r and $(1+r)\,\hat{A}\,(r)$ has a dominant root smaller than one for each r, $0 \leq r < R$. We can therefore form the mathematical series for prices $\hat{p} = p/w$ in terms of the wage-rate:

$$\begin{aligned}\hat{p}\,(r) &= (I - (1+r)\,\hat{A}\,(r)\,)^{-1}\,\hat{l}\,(r)\\ &= \hat{l}\,(r) + (1+r)\,\hat{A}\,(r)\,\hat{l}\,(r) + (1+r)^2\,[\hat{A}\,(r)]^2\,\hat{l}\,(r) + \dots\end{aligned}$$

which converges for $0 \leq r < R$ (5).

3. Interpretation of Centre Coefficients

The series derived at the end of the last section looks like a reduction to dated quantities of labour for finished goods, but what do the variable centre coefficients mean economically?

First of all, the reduction is identical with the familiar reduction to dated quantities of labour if the lifetimes of machines are all equal to one and the machines are thus indistinguishable from circulating capital.

Secondly, at $r = 0$ the reduction shows that values of finished goods are equal to a sum of 'dated' quantities of labour, whatever the lifetime of machines and their efficiency-pattern. For

$$\hat{p}\,(0) = (I - \hat{A}\,(0)\,)^{-1} = \hat{l}\,(0) + \hat{A}\,(0)\,\hat{l}\,(0) + \dots$$

where $\hat{l}_i\,(0)$ denotes the amount of labour expended on the production of one unit of good i in group i, and $\hat{a}^j{}_i\,(0)$ the total amount of finished good j used up in group i (remember $b^i{}_i\,(1) + \dots + b^i{}_i\,(T_i) = 1$). The coefficients of the series represent 'dated' inputs of labour, but only in a special sense: in order to allow the interpretation of the i-th elements of the column vector $(\hat{A}\,(0)\,)^t\,\hat{l}(0)$ as the amount of indirect labour 'now' embodied in good i and expended 't periods

(5) It has been shown (B. Schefold (1971) that dom $\hat{A}\,(r) < 1$, $0 \leq r < R$.

— 6 —

ago', one has to imagine a certain redistribution of labour between periods which can be effected at $r = 0$. (It is not worth while to work out the details of this mental exercise here).

For $r > 0$, the reduction remains simple in the extreme case of physically constant efficiency. Each process in a group belonging to a machine of constant efficiency produces $1/T_i$ units of the total output of good i (which is itself normalised to one). The coefficients of inputs of circulating capital and of labour are all equal, hence

$$\hat{l}_i(r) = \frac{\sum_{t=1}^{T_i} (1+r)^{T_i - t}\, l_i(t)}{\sum_{t=1}^{T_i} (1+r)^{T_i - t}\, b^i{}_i(t)};$$

$$\frac{1}{T_i}\hat{l}_i(r) = l_i(1) = \ldots = l_i(T_i).$$

Equally

$$\hat{a}^j{}_i(r) = \frac{\sum_{t=1}^{T_i} (1+r)^{T_i - t}\, a^j{}_i(t)}{\sum_{t=1}^{T_i} (1+r)^{T_i - t}\, b^i{}_i(t)}$$

$$\frac{1}{T_i}\hat{a}^j{}_i(r) = a^j{}_i(1) = \ldots = a^j{}_i(T_i),$$

if j is not the machine engaged in the production of good i. If m denotes the index of the new machine entering the first process of group i, and if, for the sake of simplicity, $a^m{}_i(1) = 1$, we obtain

$$\hat{a}^m{}_i(r) = \frac{(1+r)^{T_i - 1}}{\sum_{t=1}^{T_i} (1+r)^{T_i - t}\, b^i{}_i(t)}$$

$$= \frac{r\,(1+r)^{T_i - 1}}{(1+r)^{T_i} - 1}$$

which leads to the textbook depreciation formula after premultiplication by $(1 + r)$. If the rate of profit is zero, $a^m{}_i(0) = 1$, i.e. the labour value of the machine is embodied in the product in a straightforward manner. As the rate of profit rises, $\hat{a}^m{}_i(r)$ rises monotonically and tends to T_i for $r \to \infty$. The initial rise of $\hat{a}^m{}_i(r)$ is the steeper the longer the lifetime of the machine. The centre coefficients which do not correspond to machines remain constant; only the depreciation

— 7 —

quotas rise. If the maximum rate of profit of the system as a whole is sufficiently great, the depreciation quotas approach T_i at high rates of profit, i.e. the mere rise in the rate of profit increases them to such an extent that 'costs' rise as if a new machine had to be bought for every process in the group and not only for the first.

In the general case, when the efficiency of the machine varies, all centre coefficients may vary with the rate of profit. These variations are independent of the standard of prices chosen and they can cause movements of relative prices which are more complicated than those engendered by the distribution of the dated labour coefficients in the single product case. It is tempting, though not quite correct, to say that the variations of the centre coefficients express the efficiency of the machines while the structure of the $\hat{A}$-matrix expresses interindustry relations.

Such an interpretation of the reduction could be supported by repeating that relative prices of finished goods as determined by the full fixed capital system are at the given rate of profit r equal to those determined by the centre considered as an imaginary single product system which may be written as

$$(1 + r)Gp + wv = p$$

where

$$g^j{}_i = \hat{a}^j{}_i(r), \; v_i = \hat{l}_i(r)$$

The centre coefficients $a^j{}_i$ for a raw material (or labour $\hat{l}_i$) are at $r = 0$ equal to the total input of the raw material (or labour) used up during the life time of a machine for the production of a unit of output of the finished good. That is to say, the centre coefficients $\hat{a}^j{}_i(0)$, $\hat{l}_i(0)$ (machines, raw materials and labour) at $r = 0$ are simply the total of each input j and labour used for a unit output in the i-th integrated process. The centre coefficients $\hat{a}^j{}_i(r)$, $\hat{l}_i(r)$ deviate form $\hat{a}^j{}_i(0)$, $\hat{l}_i(0)$ only to the extent that the distribution of inputs is uneven over the integrated process, for if the machine is of constant efficiency and lasts T_i years, the centre coefficients of a raw material and of labour are (independently of the rate of profit) equal to T_i times over the inputs to the integrated process during any one year. (They are the same per unit or output). Only those centre coefficients differ at $r > 0$ a great deal from centre coefficients at $r = 0$ which represent machines or spare parts, because the input of the machine itself and of the spare parts are naturally the most unevenly distributed: the input of a machine (and often also of a spare part) occurs only once during the integrated process.

The interpretation of the centre as an imaginary single product system is of importance not only because it helps to visualize the generalized reduction to dated quantities of labour. It is also interesting in itself, because it proves that (provided wear and tear are calculated correctly, i.e. provided, the centre coefficients are

— 8 —

known) we can intuitively deal with a fixed capital system as with a single product system, if the rate of profit is given and fixed. Moreover, we find that the rate of profit is not only equal to net income minus wages over total capital, it is also equal to net income minus wages in the centre over capital employed in the centre, i.e. net profits over total capital including the entire stock of machines are equal to net profits over total raw materials used up during the integrated process (corrected with a factor dependent on r and expressing the unevenness of the inputs during the integrated process) *plus* annual wear and tear of the machine as expressed by the proper amortisation coefficient. In formulas we get from the price equations (written as a system of joint products (6) which contains the old machines in explicit form — $\hat{\hat{p}}$ is the corresponding price vector)

$$(1 + r)\, A\hat{\hat{p}} + l = B\hat{\hat{p}}\,;$$

and for the centre

$$(1 + r)\, \hat{A}\,(r)\, \hat{p} + \hat{l}\,(r) = \hat{p}\,,$$

for any not vanishing pair of activity levels q (activity levels for the joint production system) and q (activity levels for the centre) two formulas for the rate of profit: either

$$r = \frac{\bar{q}\,(B - A)\,\hat{\hat{p}} - \bar{q}l}{\bar{q}A\hat{\hat{p}}}$$

or

$$r = \frac{q\,(I - \hat{A})\,\hat{p} - q\hat{l}}{q\hat{A}\hat{p}}\,.$$

This means that Marx was not far off the mark when he treated constant capital as a flow. The two formulas show that net profits per period over the total stock of capital are equal to net profits per period over the circulating capital used up per period in the centre as an imaginary single product system. (Activity levels in the centre system can be related to those in the original system in a meaningful way, but we shall not discuss this matter.)

---

(6) $A = \begin{pmatrix} a_1\,(1)\,,\ m_1\,(0) \\ \dots\dots\dots \\ a_f\,(T_f)\,,\ m_f\,(T_f - 1) \end{pmatrix}$, $\hat{\hat{p}}' = (p_1 \dots, p_f\,,\ p_{m,1}\,(1)\,, \dots, \dots, p_{m,f}\,(T_f - 1)\,)$, etc.

— 9 —

4. Further Applications

We now consider the effect of the uneven distribution of inputs during the integrated process.

As an example, take the case where finished good i is produced by a machine (finished good No. 1) which lasts for two years so that

$$\hat{a}^j{}_i(r) = \frac{(1+r)\, a^j{}_i(1) + a^j{}_i(2)}{(1+r)\, b^i{}_i(1) + b^i{}_i(2)}, \qquad j = 2, \dots, f$$

and

$$\hat{l}_i(r) = \frac{(1+r)\, l_i(1) + l_i(2)}{(1+r)\, b^i{}_i(1) + b^i{}_i(2)}.$$

These functions will *fall* as r rises, if

$$\frac{a^j{}_i(1)}{a^j{}_i(2)} < \frac{b^i{}_i(1)}{b^i{}_i(2)} \qquad j = 2, \dots, f$$

and

$$\frac{l_i(1)}{l_i(2)} < \frac{b^i{}_i(1)}{b^i{}_i(2)}$$

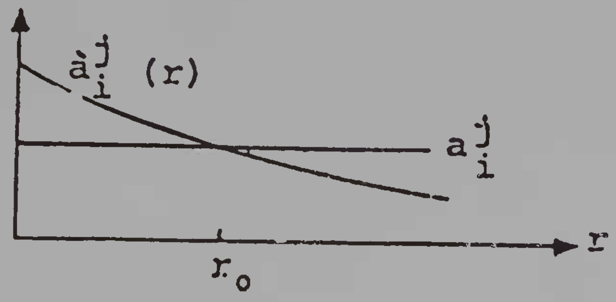
which is the case if the machine is of 'falling physical efficiency' (7). The machine produces less and yet requires more inputs as it grows older.

If $\hat{a}^j{}_i(r)$, $j = 2, \dots, f$, and $\hat{l}_i(r)$ fall sufficiently fast, the price may fall in those ranges of the rate of profit where the prices of the inputs rise only moderately or fall with r, but, as has been stressed above, a falling price in terms of the wage rate indicates that a truncation of the system would be advantageous in the corresponding range of the rate of profit.

As a further application of the reduction to dated quantities of labour, let us note a case of switching and reswitching which is connected with the variable efficiency of machines. Suppose for instance that a process producing good i by means of circulating capital a_i and labour l_i could be replaced by a group of two processes $a_i(1)$, $l_i(1)$ and $a_i(2)$, $l_i(2)$ using a machine which already exists in the system and is of falling efficiency. Suppose for simplicity that $a^j{}_i(1)$, $a^j{}_i(2)$ are positive if and only if $a^j{}_i$ are positive, $j = 2, \dots, f$, and that $a^1{}_i = 0$. The curves for $\hat{a}^j{}_i(r)$, $\hat{l}_i(r)$ could then be of the

(7) See Schefold (1974).

following type:



In the graph, the function $\hat{a}^j_i(r)$ expressing the new process cuts the constant a^j_i which corresponds to the inputs used in the old process at $r = r_0$. If this happened by coincidence for all inputs $a^j_i, j = 2, \dots, f$, and l_i simultaneously, one should expect the system as a whole to have a switch in the neighbourhood of r_0 if $\hat{a}^1_i(r)$ does not exert to great an influence at low rates of profit. Since $\hat{a}^1_i(r)$ rises monotonically it is conceivable that a reswitch occurs before the maximum rate of profit is reached. It goes without saying that the pattern of switches increases in complexity if the efficiency of the machine is rising in respect to some inputs and falling in respect to others. However, it is the point of the construction of the centre and the reduction that they make the considerable complications arising from the efficiency pattern of machines more transparent.

One might object to this approach that depreciation formulas in real life do hardly ever take account of complicated efficiency patterns and that we are therefore seeking to reach a degree of precision which has no relevance, except in so far as we satisfy ourselves that we are consistent [8]. However, I believe that real life does involve more complicated efficiency patterns, but under a disguised form. Note first that if physical efficiency falls moderately with r, the functions $\hat{a}^j_i(r)$, $\hat{l}_i(r)$ will tend to fall slowly and may just about offset the rising tendency of $\hat{a}^1_i(r)$.

As a consequence, 'linear' depreciation as practised by most firms may be quite rational. The theorist must consider more complicated efficiency patterns not only because they are logically possible, but also because they arise frequently in the context of plants or machines which are not in operation but under construction or because they are old and have to undergo substantial repairs to be kept operating.

---

[8] It is true, moreover, that depreciation in real life is affected by obsolescence (which we could take into account by introducing quasirents (see Sraffa (1960), paragraph 91), and by uncertain expectations which cannot be taken into account in the present framework.

— 11 —

Such qualitative changes in the use of the machine induce the practical entrepreneur to treat the machine as a different good in the three most typical stages which all machines undergo: construction, normal operation, operation with frequent repairs. The entrepreneur will be inclined to apply linear depreciation rules only to the middle stage where they are in fact quite suitable.

We are used to apply the term 'depreciation' only to the middle stage which is probably the longest in most cases and the only one of practical relevance in many. But if we think of 'machines' which really last long, such as a ship, there is a meaningful price of production to the ship under construction (it rises as construction goes on so that 'depreciation' is negative), and there are prices of old ships at each age. Eventually the ship may still be used towards the end of its life when frequent repairs are necessary, although it is then perhaps classed in a different category. Prices are different and follow different rules in each stage so that the commodity appears to undergo a qualitative change as it leaves one stage and enters the next. Our analysis reflects these qualitative changes in the form of different patterns of price movements.

Note also the following: if the 'net output' $Y_i(t)$ of a machine used in group i in year t (proceeds from selling the finished good *minus* the costs including profits of circulating capital and wages in the same year t) is negative at first, then rises to a positive maximum and eventually falls off again, and if all prices $p_{m,i}(t)$ in each year t of the machine are positive, they will also rise at first and fall later; but the maximum of the price of the machine will be reached before its net output reaches a maximum. If the latter maximum is 'flat', prices will be highest at a time such that $p_{m,i}(t-1)$ is approximately equal to $p_{m,i}(t)$. Since we must have for $1 < t < T_i$

$$Y_i(t) = b^i_i(t)\,p_i - (1+r)\sum_{j=1}^{m} a^j_i(t)\,p_j - wl_i(t) =$$
$$= (1+r)\,p_{m,i}(t-1) - p_{m,i}(t)$$

therefore

$$p_{m,i}(t) - p_{m,i}(t-1) = r\,p_{m,i}(t) - Y_i(t)$$

a maximum with $p_{m,i}(t) = p_{m,i}(t-1)$ will be characterised by

$$r = \frac{Y_i(t)}{p_{m,i}(t)}\,.$$

The relationship shows that the machine's prices are stationary (here at their maximum) when the ratio of net output to price is equal to the rate of profit. If the rate of profit is zero, the machine has its maximum value at the point where the income generated by the machine turns positive. What we here have called 'net output' is equal to 'depreciation', i.e. to $p_{m,i}(t-1) - p_{m,i}(t)$, if $r = 0$. If the rate of profit rises, net output is equal to depreciation *plus* the financial charge $r\,p_{m,i}(t-1)$, and the maximum of net output will *ceteris paribus* shift to older ages. Thus, a male slave in the American South was most expensive towards the age of thirty years and not of twenty. (See Fogel (1974); p. 72). The example may be shocking, but none is better for a gradual change of efficiency.

5. Roundabout Processes

The considerations above suggest that it would be at least as difficult to construct an aggregate of 'capital' in fixed capital systems as in single product systems. Even more than in the single product case one seems to be bereft of any general results about capital theory which would allow to determine the character of the 'substitutions' which are supposed to take place in response to changes in the rate of profit.

The complexity of the structure of production increases as the analysis approaches the case of general joint production. Within the framework of fixed capital (which is a case in between) we now consider roundabour processes, i.e. processes where raw materials and machines that are already being produced in the existing system are being used to construct a new machine which, in conjunction with raw materials, replaces an existing process.

Such roundabout processes are already contained in our analysis. To see this, suppose process 1 which is a circulating capital process a_1, l_1 (in vector notation) is to be replaced by a roundabout process; i.e. a new process o is introduced which employs raw materials $a_0 = (a^1{}_0, \ldots, a^f{}_0)$ and labour l_0 to produce one unit of a machine, finished good 0 with price p_0 that is in turn used in an integrated process with lifetime T_1 to produce good 1. The new system is assumed to be productive and to satisfy all other assumptions of fixed capital systems.

The point is now that the new processes can be taken together to form *one* roundabout process which looks like an integrated process where nothing is produced in the first year. For the $T_1 + 1$ processes (in vector notation)

$$(1 + r)\, a_0 p + w l_0 = b_0 p = p_0$$

$$(1 + r)\,(p_0 + a_1(1)\, p) + w l_1(1) = b_1(1)\, p + m_1(1)\, p_{m,i}(1)$$

$$(1 + r)\,(a_1(2)\, p + m_1(1)\, p_{m,1}(1)) + w l_1(2) = b_1(2)\, p + m_1(2)\, p_{m,1}(2)$$

$$\ldots$$

$$(1 + r)\,(a_1(T_1)\, p + m_1(T_1 - 1)\, p_{m,1}(T_1 - 1)) + w l_1(T_1) = b_1(T_1)\, p$$

— 13 —

can be combined into one integrated process where t runs from 0 to T_1, and where the centre coefficients are

$$\hat{a}^j_1(r) = \frac{\sum_{t=0}^{T_1} (1+r)^{T_1-t} a^j_i(t)}{\sum_{t=0}^{T_1} (1+r)^{T_1-t} b^1_1(t)}, \qquad j = 1, \dots, f,$$

$$a_1(0) = a_0, \text{ etc.}$$

Thus, a roundabout process is equivalent to the group of processes of a machine under construction like a ship which takes a year to be built and then runs for several years. (The analysis is at once extended to the case of a construction period of more than one year.)

Nobody denies that technical progress entails the use of 'roundabout' processes. The question is whether neoclassicals were right in asserting that roundabout processes will be introduced in response to a falling rate of profit and a rise in wages, and that this introduction of roundabout processes corresponds to a rise of the capital labour ratio.

On purely logical grounds, there is nothing to substantiate this claim if roundabout processes are defined in the abstract fashion suggested above. We have already indicated that the introduction of machines can produce a number of switches. Only if specific assumptions about the technological character of 'increasing roundaboutness' are made, does it become possible to construct a hierarchy of techniques according to 'technological development' that corresponds at the same time to the hierarchy of capital-labour ratios. This is discussed in Schefold (1976, 1979) where it is shown thata more mechanized (more 'roundabout') technique entails a higher capital-labour ratio, if (broadly speaking) the amount of raw materials used up in each year is not diminished by the introduction of the machine, and if the efficiency of the machine does not fall fast.

These are special assumptions, and in order to guard oneself from drawing too generous conclusions it may be useful to consider an example where the total sum of labour employed is greater and the total sum of each input of finished goods used is smaller for a roundabout process than for the one it replaces, and where the former is all the same not more labour intensive than the latter. The example shows also that it would be wrong to think the assumption of a diminishing use of raw materials sufficient to ensure that the maximum rate of profit rises.

Consider the familiar corn-corn economy (p_c is the price of corn in terms of the wage rate):

$$(1+r)\, a p_c + l = p_c .$$

— 14 —

Suppose, a machine with price p_0 is invented, produced by means of corn and labour and used for two years to produce corn. The price of the one year old machine is p_1 (both prices in terms of the wage rate):

$$(1 + r)\, a_0 p_c + l_0 = p_0$$
$$(1 + r)\, (a_1 p_c + p_0) + l_1 = b_1 p_c + p_1$$
$$(1 + r)\, (a_2 p_c + p_1) + l_2 = b_2 p_c\,.$$

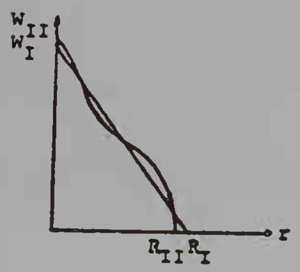
Of course,

$$a_0 + a_1 + a_2 < b_1 + b_2 = 1\,.$$

If $a = \frac{1}{2}$, $l = 1$ in the original corn-corn economy, the maximum rate of profit R_I equals one. Suppose, a little labour is diverted to produce the machine, and suppose that the machine entails reduced use of corn ($a_0 + a_1 + a_2 < a$) while $l_0 + l_1 + l_2 > l$. If the particular values $l = 1$, $a = \frac{1}{2}$, $b_1 = b_2 = \frac{1}{2}$ and

$$a_0 = \frac{31}{504} \qquad l_0 = \frac{3}{140}$$
$$a_1 = \frac{1}{4} \qquad l_1 = 1$$
$$a_2 = \frac{2}{315} \qquad l_2 = \frac{1}{3}$$

are chosen, there is a new maximum rate of profit R_{II} such that p_1 is positive for $0 \leq r < R_{II}$. There are no less than *three* switch points at $r = \frac{1}{4}$, $r = \frac{1}{2}$, and $r = \frac{3}{4}$ where the price of corn in terms of labour commanded is the same in both systems. Contrary to what one might expect at first sight, the roundabout process is more profitable at $r = 0$ and $R_{II} < R_I$ although the total sum of raw materials used is diminished. The capital labour ratio (calculated e.g. under the Golden-Rule condition where the rate of profit is equal to the rate of growth) fluctuates accordingly.



(Intersection between wage-curves for two techniques; numerical values are given in the text.)

— 15 —

The possibility of three switchpoints in such a simple example looks disturbing. It confirms not only the now well-known critique of the neoclassical production function but indicates that any meaningful notion of a 'degree of mechanization' such as might be of use in a theory of planning will be difficult to define. Since the three processes of the new technique involving the machine represent *one* roundabout process, the example also shows that the uneven distribution of inputs and outputs during its duration is responsible for the multiplicity of switchpoints; if the alternative technique consisted only of one process of the corn-corn variety, the corresponding wage-curve would be linear, and at most one switchpoint could ensue. This conclusion could be reenforced by an analysis of the centre coefficients corresponding to the roundabout process.

To summarize: We have shown that fixed capital systems allow a generalization of the reduction to dated quantities of labour which is based on the centre as an imaginary single product system incorporating the effects of the age-dependent variations in the efficiency of machines. The centre thus expresses the idea that fixed capital can be treated as a flow after depreciation coefficients have been calculated. Variable efficiency patterns of machines generate centre coefficients which may be complicated functions of the rate of profit. They therefore give rise to switchpoints which would not occur if machines were of constant efficiency or if they lasted only one year like circulating capital. Finally it has been shown that the old concept of roundabout processes can be formalized in fixed capital systems; an example was used to prove that diminished use of raw materials accompanied by an increased use of labour does not necessarily imply that the introduction of the roundabout process implies a reduction of capital intensity.

REFERENCES

R. W. Fogel, S. L. Engerman [1974]: Time on the Cross, London.

L. L. Pasinetti [1973]: *The Notion of Vertical Integration*, in *Economic Analisis*, « Metroeconomica », XXV.

B. Schefold [1974]: *Fixed Capital as a Joint Product and the Analysis of Accumulation with Different Forms of Technical Progress, mimeograph*, Italian Translation is « Contributi alla teoria della produzione congiunta », a cura di L. Pasinetti, Bologna 1977.

— — [1971]: « Theorie der Kuppelproduktion », Basel (private print).

— — [1976]: *Relative Prices as a Function of the Rate of Profit: A Mathematical Note*, in « Zeitschrift für Nationalökonomie », 36, pp. 21-48.

— — [1976]: *Different Forms of Technical Progress*, « The Economic Journal » 86, pp. 806-819.

— — [1975]: *Capital, Growth and Definitions of Technical Progress*, to appear in test schrift for G. Bombach, « Kyklos » 32.

P. Sraffa [1960]: *Production of Commodities by Means of Commodities*, Cambridge.

[7]

Sraffa and Applied Economics: Joint Production

Bertram Schefold*

> "In these circumstances there will be room for a second, parallel process which will produce the two commodities by a different method..."[1].

1. THE CLASSICAL METHOD

Is there a field of application for Sraffa's theory? There can be no doubt that Sraffa's work has so far almost exclusively been interpreted as a critique of the theory of value. As a consequence of discussions which were decisively influenced by Sraffa, the economic profession has in the last 15 years begun to accept the fact that the aggregate production function is an illegitimate tool of analysis. Wider implications for the neoclassical school are still being debated. Within the classical approach, the traditional form of the labour theory of value was superseded. But the empirical applications were few; they really consist only of some attempts to calculate wage curves. These attempts were necessarily inconclusive as far as the debate about capital theory was concerned while they did shed some light on the analysis of technical progress[2].

* I should like to thank P. Garegnani for valuable and extensive comments on an earlier version of this article.

[1] P. SRAFFA, *Production of Commodities by Means of Commodities*, Cambridge, CUP, 1960, p. 43.

[2] G. MARZI and P. VARRI have calculated wage curves from input-output data of the Italian economy. W. KRELLE has done the same for the Federal Republic of Germany (G. MARZI-P. VARRI, *Variazioni di produttività nell'economia italiana: 1959-1967*, Bologna, Il Mulino, 1977; W. KRELLE, "Basic Facts in Capital Theory. Some Lessons from the Controversy in Capital Theory", *Revue d'Economie Politique*, 1977, vol. 87, pp. 282-329). Such wage curves, if calculated and compared for different time periods, do not prove anything about the switch points between wage curves belonging to different technologies which compete in a given moment of time, but they do give indications as to the prevalent form of technical progress which went on between periods. According to Marzi and Varri the maximum rate of profit seems to have fallen in Italy as if mechanization (see B. SCHEFOLD, "Different Forms of Technical Progress", *The Economic Journal*, vol. 86, 1976, pp. 806-819) had increased but there is no presumption to expect this result to hold in general.

I believe that Sraffa's theory has a wider scope and can provide a framework for much of modern applied economics. This does not imply a universal transformation of the field, since the methods employed by good applied economists can sometimes be justified better on the basis of classical rather than neoclassical theories while both predict similar results.

Input-output analysis provides a good example of an important tool of applied economics based on a classical methodology. Although Leontief has taken pains to emphasize the compatibility of his conception with the neoclassical tradition[3], it is quite clear that it fits in much better with the classical. For it is the point of input-output analysis to regard the methods of production in use as given independently of relative prices, to derive conclusions from the technological interdependence and to consider the influence of changes in relative prices and in technical progress on the coefficients of the input-output structure only subsequently.

It is a fundamental principle of classical economics that it *separates* a) the determination of outputs, b) the determination of distribution and c) the analysis of the relations between the distributive variables and between them and relative prices. It is this which makes the classical approach a better basis for applied economics. It provides direct links between the essential magnitudes in the system (the "short chains of reasoning", which Marshall was looking for)[4], while the countless relations of interdependence in a general equilibrium system are a poor guide to any application. Input-output systems also serve to analyse "interdependence", but by treating the methods of production in use as given and the determination of prices and distribution as separate issues, the analysis of the dependence of activity levels on final output becomes manageable even if feedbacks between different industries have to be taken into account.

The strength of the classical approach is most visible in dynamic analysis. While it is hard to represent the evolution of the economic system even without technological change as a sequence of Walrasian equilibria empirically, there are now several quite successful large econometric models which capture the process of macroeconomic dynamics and of structural evolution by means of a combination of an input-output system for the representation of technology with a macroeconomic model for the representation of the evolution of effective demand in its interaction with distribution and government policy, and a demand model based on *aggregate* demand functions which may be differentiated according to socio-economic criteria. My education in classical

[3] W. W. Leontief, *The Structure of American Economy 1919-1935* (1941), sec. ed. 1951, White Plains, International Arts and Sciences Press, repr. 1976, p. 37.

[4] P. Garegnani, "The Classical Theory of Wages and the Role of Demand Schedules in the Determination of Relative Prices", *American Economic Review*, May 1983, pp. 309-313.

economics proved very useful for the understanding of the true functioning of these models which one encounters in research on the economics of energy. However, one must admit that the eclectic character of most econometric models does not allow an unambiguous interpretation of their theoretical background.

Input-output analysis is not the only area which could benefit from an interpretation along classical lines, but research into the empirical usefulness of modern classical theory has, among other things, been impeded because its proponents have tended to focus on the critique of the neoclassical theory of capital and distribution instead of on positive contributions. The present paper is concerned with some preliminary considerations which might lead towards an application of Sraffa's theory of joint production by discussing joint production input and output tables and by analysing the meaning of a possible underdeterminacy or overdeterminacy in the system if the number of processes used is not equal to the number of commodities (goods with positive prices) produced.

Other tools of the classical theory which had been used by the classical economists themselves and could also be applied by modern economists are not being discussed here but a parallel paper will deal with the classical analogue of the Marshallian supply curve.

We assume that distribution determines a uniform rate of profit (alternatively: a hierarchy of rates of profit). Demand for consumer goods is treated very simply as emanating from given social needs. The relationships between those needs are thought to reflect complementarity rather than substitutability. The needs evolve with the growth of wealth in different segments of the population, but corresponding Engel curves do not have to be considered, since we are dealing with a given long period position. We assume, however, that there may be different domestic processes of production to fulfill the same needs, and that their choice depends (if we abstract from habits, taste, ignorance, etc.) on the cost of providing the corresponding services, hence on relative prices. Goods which are close substitutes (where the rise in the price of one leads to an increase in the consumption of the other because they fulfill the same need) are then to be treated as the same commodity if they can only be distinguished according to taste and not as alternative means used in different methods of production for the fulfillment of the same need. This manner of treating demand has been successful e. g. in the explanation of changes in energy consumption. It relates to the classical view of the matter and will turn out to be helpful for the understanding of Sraffa's theory of joint production.

2. JOINT PRODUCTION AND ACCOUNTING

There seems to be an unsurmountable gap between the treatment of joint production in economic theory and in the literature on business administration. Models of general equilibrium and the von Neumann model determine prices of joint products but this determination does not seem to provide definite rules for those working in the field of business administration, for the latter regard the theory of prices of joint products as largely indeterminate and discuss "practical" rules for the setting of prices in diverse circumstances. Of course, no theory can be expected to provide a ready-made set of rules for pricing, given the complexity of every-day life. But one can show why classical theory may serve as a background to explain some accounting procedures and why these procedures fail in specific cases; the "gap" may thus be bridged.

The first mistake of the prevailing economic theory, most clearly visible in the von Neumann model, consists in the assumption that a definite complete list of the goods to be used and produced by any method of production can be established for each process independently of the others. If this were the case, few environmental problems would arise. We do not have *complete* knowledge of what the smoke of factories consists of; far less do we know about the synergetic external effects of different processes of production. In reality, the identification of those goods which are to be the objects of economic planning is the first step in the practical transformation of the material world which we call "production". The goods so selected are potential commodities; everything else is ignored until external effects are being felt. Sraffa's analysis of joint production therefore starts from the system of commodities and processes which are actually used and considers the use of alternative processes and the introduction of new goods only subsequently and one by one.

All production is joint production as far as "goods" are concerned. The traditional emphasis on single production of commodities in economic theorizing is not simply due to the fact that the theory of single product industries is much simpler than that of joint production; rather, it reflects economic practice according to which production originally is a purposeful activity, in general directed at obtaining *one* specific good which may be sold as a commodity. Goods which are produced jointly are usually turned into commodities only later in order to increase the profitability of the process. Although there are exceptions to this rule, e. g. in transportation, where any vehicle is introduced to carry many products, it is surprising how often one finds a single purpose at the origin of what later becomes a multiproduct industry. Even the ships which produced trips from Spain to the West Indies and back jointly were first used only to import treasure, not to export cloth.

The economic growth of industries seems to be characterized similarly by an initial disregard for joint costs; e. g. the infrastructure is often taken care of properly only after the industries have been set up. This is also the case for the joint costs of a national economy and even for those of the world which increase faster than our recognition of the global interdependence of many environmental problems. The reason is that the planning process starts with a simplified view of the world — a simplification which often implies violence.

The representation of joint production in economic models ignores this dynamic element. In consequence, the treatment of joint production in general economics is closed, but it does not lead to a view of the sequence of events in the evolution of joint production processes. By the same token, it lacks specificity; one does not find a morphology of joint production in economic theory.

The opposite picture emerges at the level of the theory of the enterprise. There does not seem to be a generally accepted theory for the determination of prices in multiple product industries in the field of business administration, but there are essentially two approaches; diverse variants are discussed for different applications of each to different industries in the literature. *Either*, one tries to *set prices*. To this end, it is thought necessary to ascribe costs to individual products by means of splitting up overheads, depreciation etc. according to various rules, and to add a normal profit in accordance with a target rate of return or some similar notion. According to this theory of full cost pricing in its several variants, firms are able to administer prices in imperfect markets freely, but within limits, so as to guarantee a satisfactory return at normal levels of capacity utilization. The principal drive of competition ought to be expressed not in higgling about prices but design, marketing, product innovation etc.[5].

Alternatively, it is being thought that market prices are given within a narrow range even in imperfect markets and under modern conditions, because there is competition between products which are close substitutes and because the entrepreneurs almost always have some notion as to what the traffic will bear. One then asks *where profits are made*, i. e. how the sales proceeds contribute towards the covering of the expenses and the profits of the various decision taking units within the firm so as to obtain a measure of their efficiency and a guide for future investment policies. Methods for ascribing profit contributions are again diverse. It is sometimes thought best to ascribe costs to that unit of an enterprise where these costs appear as direct costs. For e. g. the overheads of the division responsible for sales may be direct costs to the central management of the firm. One can thus go some way towards the reduction of overheads to

[5] See J. M. Blair, *Economic Concentration; Structure, Behavior and Public Policy*, New York, Harcourt Brace Jovanovich Inc., 1972.

direct costs and restore simple rules for profitability, but the limits of the approach are clearly visible, e. g. in the case of the overheads due to the intertemporal use of fixed capital. It is sometimes being said that business accounting is an art rather than a science[6].

But it is clear that, on Sraffa's assumptions, prices *are* determined. The discrepancy between the uncertainty as to the proper rules of accounting and the uniqueness of prices in theoretical systems requires an explanation.

First of all, it may be shown that the main basic rules of accounting all come to the same thing and are consistent with the theoretical solution based on long run prices, provided there is an equilibrium. Consistency then means that if the accounting rules are applied to individual or groups of processes in a Sraffa system, they imply accounting prices and ascriptions of costs which do not lead to other prices than the Sraffa prices themselves. Some examples may suffice to show this.

If it is being asked how the costs of two joint products are to be ascribed to the individual products at given market prices, the answer simply is that, looking at the process where the products are being produced jointly, any splitting up of the costs will be consistent with Sraffa prices, provided the market prices are equal to Sraffa prices, and the total costs (including normal profits) are equal to the sales proceeds. The arbitrary rule for splitting up the costs (e. g. according to weight or calorific value) is simply irrelevant to the "macroeconomic" determination of prices. If the same two products happen again to be produced jointly in another industry in different proportions as Sraffa suggests, a different ascription of costs according to any such rule will lead to a different result in the other industry, but the discrepancy does not matter since the purpose of the accounting procedure is only to provide a basis for calculation for the price to be set or the profit contribution out of equilibrium while we suppose that equilibrium prices have been determined "behind the back of the producer". (If the two processes were used within the same firm, a different accounting procedure would be used).

Alternatively, there is sometimes a distinction being made between main products and subsidiary products. It is being asked what the correct pricing of the subsidiary product is, given the prices of the main product. The price of the total output of the subsidiary product is calculated by deducting the proceeds from selling the output of the main product from the total joint cost of production. The result will be the Sraffa price of the subsidiary product in equilibrium conditions. In disequilibrium conditions, the calculated price of the subsidiary product may be anything and even negative, for if e. g. the price of the main product is very high, its cost

[6] E. S. Schmalenbach.

of production will be more than covered even if the subsidiary product is given away free.

Thirdly, the "marginal method" may be mentioned. Here it is being asked what the prices of two products should be if they can be produced in varying proportions. The rate of transformation is said to define the relative prices of the outputs. But this is equivalent to a calculation of prices, given a shadow rate of interest, by considering the equation of the process in actual use and one resulting from a (small and continuous) variation of the inputs and outputs at the margin. It is clear that one thus obtains two Sraffa processes — a result which is consistent with Sraffa prices on the assumption that the shadow rate of interest equals the general rate of profit.

In this way, some of the accounting rules may be rationalized on the basis of Sraffa's theory of prices in joint production systems: the pricing rules lead to "equilibrium" prices under "equilibrium" conditions. This is in itself not surprising. The explanation is in fact of interest only to the extent that the underlying theory is convincing. That this is the case will be argued by illustrating its explanatory power first in cases where it applies directly, second in cases, where there is a "disequilibrium" and where it has to be shown how an equilibrium is established in a real process of adaptation.

3. WHY "SQUARE" JOINT PRODUCTION SYSTEMS?

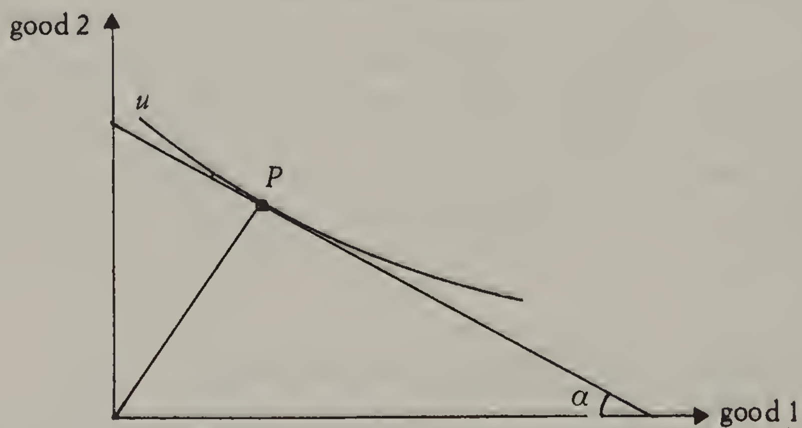
The strange and crucial assumption which allows one to determine the equilibrium seems to be that the number of commodities should be equal to that of processes. Economic theorists are sometimes puzzled by this because they do not know that this assumption will be fulfilled for the processes actually used and the commodities sold not only in Sraffa systems but also in von Neumann type systems with probability one, as I have shown elsewhere[7]. The proof is based on the assumption that a vector of final demand, i. e. the composition of output, is given. It is then shown that among all von Neumann systems capable of producing that vector at a given rate of growth (equal to the rate of interest) a system will be chosen for which the equality of the number of commodities with positive prices is equal to that of processes used except for a set of systems which is of measure zero in the set of all possible von Neumann systems of a given order.

The accountants and applied economists do not encounter an equality

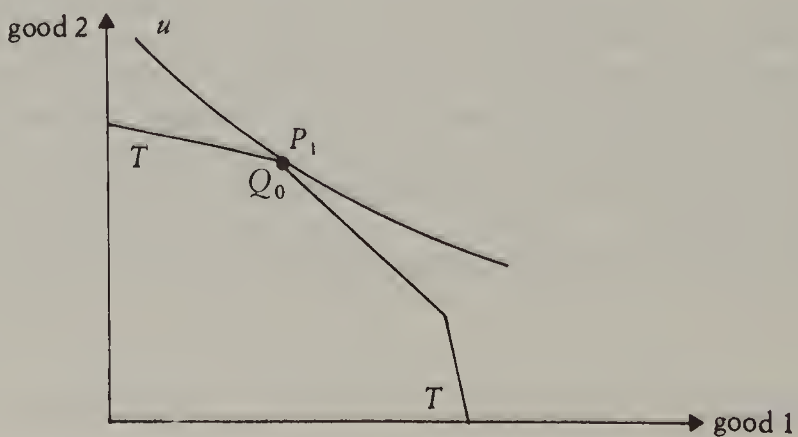
[7] B. SCHEFOLD, "On Counting Equations", *Zeitschrift für Nationalökonomie,* vol. 38, Heft 3-4, 1978, pp. 253-285.

of the number of processes and commodities in practice. The ordinary theorist, even if he should accept the assumption as an initial hypothesis, has never been told how these conditions reproduce themselves in the process of economic development. And none of them recognizes the usefulness of considering a system of prices of production, since it seems to be generally agreed that whatever happens to prices under changing conditions must be explained in terms of "supply and demand".

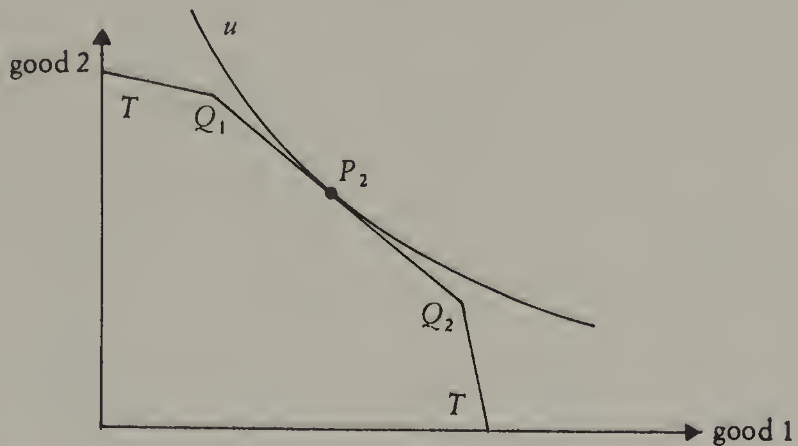
As a matter of fact, "square" systems do not necessarily result from general neoclassical assumptions. For if two commodities are being produced by one and only one process in rigid proportions, the marginal rate of substitution (as given by the slope of the indifference-curve u in the point of equilibrium P in the following diagram) will determine the relative price with $p_1/p_2 = tg\alpha$ unambiguously:



If there are, on the other hand, many potential processes of production which span the production possibility surface, there will be two cases: Either, a corner of the transformation "curve" (represented by a convex polygon T in the plane) will be the equilibrium point P_1 using *one* process Q_0:



or the equilibrium may be a point P_2 on a segment spaned by *two* processes Q_1 and Q_2:



Neither type of solution P_1, P_2, can be regarded as a fluke case since a small perturbation of the technology with given indifference curves (or of the indifference curves, given the technology) will leave the essential properties of the equilibria unchanged: there will be more commodities (two) than processes used (one) in P_1 while the number of processes used equals that of commodities (two) in P_2. The coexistence of several processes in actual use to produce the same set of commodities at positive prices is therefore not excluded and not exceptional in neoclassical theory but an excess of the number of goods with positive prices over the number of processes used is also not unlikely.

Casual observation indicates that we often encounter joint production processes yielding several commodities while additional processes producing the same commodities are not visible without close scrutiny. Common sense, the doctrines of accounting and neoclassical theory therefore all seem to contradict Sraffa's hypothesis of a necessary equality of the number of commodities and of processes used. We therefore have to ask from which assumptions the hypothesis follows and why it might be useful.

4. EXAMPLES FOR THE USE OF "SQUARE" SYSTEMS

I shall first illustrate the usefulness of the approach, taking the number of processes equal to that of commodities. Phenomena which might be classed under the general heading "supply and demand" can here be explained more specifically.

Consider, for instance, a Sraffa system in which the last process produces natural gas by means of labour and various commodities, and in which the second but last process produces coke and gas from coal by means of basics and labour. Assume that the two kinds of gas are substitutes. In such a system it may happen that — contrary to what happens in basic single product systems — a rise in the cost of gas production (due e. g. to an increase of labour requirements) leads to a *fall* of the price of coke and of all other prices if gas is a non-basic and coke a basic. The reason is, simply, that the rise in the price of gas contributes towards the expenses of the coke industry so that the price of coke may be lower and this, being the price of a basic, leads to a fall of all other prices, provided the remainder of the system works insofar like a single product system[8].

Similarly, if we take a step towards considering a situation in which the number of commodities exceeds that of processes and assume that a new consumption good results from known processes in a Sraffa system, the selling of the new commodity (which had previously been a useless by-product) will generate a revenue such that all prices fall, if the new commodity is produced in an indispensable process, but some prices may rise and some fall if the process is not indispensable. If, in the previous example, the second but last process does initially not produce any gas, the introduction of gas in that process (which is indispensable) will allow to reduce the price of coke and hence of all basics. But if a by-product emerges as a new commodity in the last process which now produces natural gas and some other non-basic, e. g. petroleum, the price of natural gas falls in the last process, the same happens in the coke-producing process, and this forces the price of coke to rise so that all other prices will rise.

Generally, the criterion for the choice of technique in the presence of joint production cannot be that of reducing individual prices in terms of the wage rate as in the case of single product systems. The reduction of the price of natural gas is directly beneficial only to gas consumers while the rise of the price of coke and all other basics affects all consumers.

These arguments rest on the assumption that a rate of profit (or, in practical applications, a hierarchy of rates of profits) may be regarded as given for the analysis of questions related to the choice of technique, e. g. in the area of energy economics. The applied economist is used to making this assumption in the appropriate context and he will not be surprised that joint production will lead to curious effects such as the ones which I

[8] The example is spelt out in numerical terms in B. SCHEFOLD, "Multiple Product Techniques with Properties of Single Product Systems", Zeitschrift für Nationalökonomie, vol. 38. Heft 1-2, 1978, pp. 29-53.

have described, provided the system of interdependent processes is small enough to be understood in intuitive terms. I should say that he then argues along the lines characteristic of classical economics. But research into the effects of changes in methods of production, long run demand conditions and distribution on interdependent joint production systems with a large number of equations is little known and it must be admitted that it has so far only rarely been undertaken with a view towards practical applications. Empirical input-output analysis has not yet been extended to take the scientific progress made by Sraffa into account.

The examples given indicate, however, that the approach might be fruitful because it allows to analyse effects of joint production on the economic system as a whole within a unified framework. Such effects cannot be understood satisfactorily by means of conventional approaches. For if the applied economist uses partial equilibrium analysis, the effects on individual industries and prices can be described but the global effect is lost whereas the use of input-output matrices in their present form is based on a preliminary elimination of joint production by means of aggregation procedures which are to some extent arbitrary and conceal specific effects such as that of a fall in one price causing all other prices to rise.

We now come to the counting of equations. It will be seen that the consideration of the "disequilibrium" conditions in which the number of equations is not equal to that of commodities provides the foundation for a better understanding of classical theory as well as of the diversity of accounting rules in disequilibrium situations.

5. COUNTING OF EQUATIONS I: THE CASE OF OVERDETERMINATION

It is easy to see what happens if the number of processes seems to exceed that of commodities. Some processes will then be more (and some less) profitable than others. Surplus profits may be consolidated as rents and accrue to those who control the causes for the permanence of the multiplicity of methods. The incomes of owners of land or of a patent are based on property rights; they can — but they need not be — identical with the entrepreneur who receives the ordinary profits. The surplus profits are temporary in the case of technical progress. Sraffa emphasized yet another case: the rents of obsolescent machines which are not being produced any more so that their capital cost need not be accounted for and the cost of production of their products reduces to that of raw materials (with normal profits) and labour.

But I want to use a *crude* example to show that one can go still further and include "domestic processes of production":

$$
\begin{aligned}
&CH \,\&\, C \,\&\, L \rightarrow WW\\
&CH \,\&\, O \,\&\, L \rightarrow WW\\
&I \,\&\, AH \,\&\, L \rightarrow RH \,\&\, WW\\
&P \quad \& \quad L \quad \rightarrow RH
\end{aligned}
$$

In the first process central heating and coal and labour are used to produce warm water in houses, in the second central heating and oil and labour are used to produce warm water in houses, in the third insulation and additional heating and labour are used to produce a renovated house and warm water and in the fourth paint and labour are used to produce a renovated house. If we assume the prices of inputs *CH* (central heating), *C* (coal), *L* (labour), *O* (oil), *I* (insulation), *AH* (additional heating), *P* (paint) to be known on the assumption that the outputs are non-basics, we have four equations and two unknowns: prices of warm water (*WW*) and room heating (*RH*).

Everybody knows that such situations of overdetermination are frequent and that they may persist for some time. Among the causes, first habit and ignorance are certainly important. It has been estimated that if the main devices for saving energy which were known, in partial use in 1975 and which would have been profitable at 1978 prices had been used generally in 1975, energy consumption would have dropped by a third[9].

Second. But it is also possible that prices of inputs rise to make prices of output match. Thus, the price of coal may rise to match the price of oil, and the rise in the price of coal may be engendered by a rise in the wages of miners. Part of their wages then has the character of a rent. A spurt of demand may drive up all the prices of materials for insulation temporarily. Such differentials are assumed to get eliminated through competition in the Ricardian long run unless the cause for the differential is permanent and the corresponding rent can be appropriated.

Third. The prices of production are centres of gravitation. All costs, including the cost of insulation, tend to get reduced to the cost of production which is assumed to be given. The point is that it then becomes possible to analyse the process of disequilibrium with reference to an equilibrium defined by prices of production and hence an equality of the number of processes and "commodities" where the latter include various objects which receive a permanent rent such as lands, patents, workers with particular talents etc. Since the formation of habits and property

[9] Deutsche Shell Aktien Gesellschaft, "Perspektive der Energieversorgung", Oktober 1980.

rights are among the causes for lasting differentials, the analytical task involves questions of political economy. One has to decide which techniques will turn out to be "socially necessary". The accounting rules will reflect differences of the institutional set-up and thus conceal the fundamental similarity of different phenomena of overdetermination. There is again an equality between the number of positive prices and the number of processes in the pure case of Ricardian rent, if different kinds of land with positive rents are counted as so many "commodities".

6. COUNTING OF EQUATIONS II: THE CASE OF UNDERDETERMINATION

The converse case where there seem to be not enough processes is perhaps the more interesting. There can be no doubt that there are many joint production processes in industry with little or no possibility for a variation in the proportion of the outputs produced, and without additional processes being visible which might help to determine the prices of production simultaneously with the first according to the rule of counting of equations. Here, neoclassical tradition as well as the textbooks on business administration suggest that we rely on "demand" but Sraffa argues that, in such cases, conditions of "demand" will generally ensure that further processes will be used which are distinguished from the first by different proportions in which the commodities in question are used as outputs *or as inputs*, for otherwise the commodities could not be produced and used in the combination socially required.

In the simplest case the underdeterminacy of the price system is made to disappear by letting superfluous commodities disappear. If a main product cannot be produced as a commodity without also producing some by-product in excess of the demand from other producers or from consumers, the by-product cannot be a commodity with a positive price; hence it is not part of the system and does not cause an underdeterminacy. (In applied theory, it is not asserted that goods are free if and only if they are overproduced. In particular, if the by-product is a waste which must be removed at some cost, the cost of its removal has to be regarded as an *input* to the production of the main product. In either case there is no difficulty to the theory of price formation).

There can be no question, however, that the indeterminacy of "too few" processes being present does arise in a less trivial manner in phases of transition. The difficulty seems to be the greatest if the proportion in which joint outputs are being produced cannot be varied, i. e. if there is "rigid" joint production. Let us consider an example of such a disequilibrium: Nuclear power stations cannot vary their output of electricity quickly for technical reasons. The addition of a nuclear power station creating an adequate supply of electricity during daytime therefore leads to an excessive supply of electricity at night. As a result, the market price for electricity produced at night will fall. The classical approach rules out as irrelevant attempts to determine the extent to which the price of excess electricity will fall by means of considering the subjective element of demand in isolation. The clue to the problem is found in the observation that the fall of the price creates an opportunity for introducing new processes which use that electricity (otherwise the price might fall to an indefinitely low level). An example is storage heating which uses electricity produced at night to generate heat during the day which may also be produced directly by other means, e. g. central heating.

If these new processes are not sufficient to lead to a match of supply and demand, a process will be required which ensures that the correct proportion according to social needs is reached. In the circumstances, it is the direct transformation of electricity produced at night into electricity produced during the day by means of stations which use electricity produced at night to pump water, and this in turn is used to produce peak load electricity during day time.

We thus have two essentially different solutions to solve the indeterminacy in the case of rigid joint production. Symbolically, the first solution looks as follows:

$$\begin{aligned} NPS &\rightarrow ED \,\&\, EN \\ EN &\rightarrow H \\ CH &\rightarrow H \end{aligned}$$

The nuclear power station (*NPS*) produces electricity during the day and electricity during the night (*ED* & *EN*); electricity during the night is used to produce heat (*H*). On the assumption that the cost of *NPS* is known and that there is an alternative process which produces *H* and determines its price at given costs (central heating *CH*), the prices of *EN* and *ED* are determined by the first two equations. It turns out that one of the outputs has its price determined as an input.

The more direct and more elegant second solution is provided if there is a separate process transforming *EN* into *ED* by means of pumping stations:

$$\begin{aligned} NPS &\rightarrow ED \,\&\, EN \\ EN &\rightarrow ED \end{aligned}$$

For simplicity, additional inputs of known costs such as labour are not shown. Here, one output can be transformed into the other.

The indeterminacy may therefore be solved on the input side because the outputs have alternative uses as inputs in other processes. Or it may be solved because a process links inputs and outputs directly as in $EN \rightarrow ED$. Or (this is the third and most conventional solution), there may be a second process which produces one (or both) of the commodities as an output. E. g.:

$$NPS \rightarrow ED \; \& \; EN$$
$$C \rightarrow ED$$

Here, electricity during day time is produced by means of old coal-fired power stations which are still used to supplant nuclear electricity generation for peak-load production. In conventional terminology, *EN* would then be regarded as a by-product in the first process and its price would be explained accordingly.

Finally, there may be a case specially emphasized by Sraffa: a second process produces positive amounts of both commodities in different proportions. Examples of this with *rigid* joint production are perhaps not common, but there may be a variability of outputs which leads to the same result, as it turns out, since small variations of output which match demand may be represented by a linear superposition of two neighbouring processes.

The last possibility is illustrated by power stations which produce hot water and electricity jointly and where there is some substitutability between outputs. For instance, coal may be used in fluidized bed combustion (*FBC*) to produce hot water (*HW*) and electricity (*E*):

$$FBC \rightarrow \mathrm{HW} \; \& \; E$$

Since the proportions are variable, we may imagine that two processes are used which differ slightly so as to satisfy demand:

$$FBC_1 \rightarrow HW_1 \; \& \; E_1$$
$$FBC_2 \rightarrow HW_2 \; \& \; E_2$$

In the limit, the two processes may fuse into one and one obtains the usual marginal condition.

The last solution seems to lead back to Marshall and the neoclassical method where a substitutability of the outputs which are being produced jointly is assumed. Demand as derived from utility determines relative prices and outputs. Whilst some influence of preferences cannot be denied, the emphasis of the classical approach is different, however, because preferences are not the only social force which are admitted as influencing the composition of output. The concrete examples which I have chosen illustrate the political element in the determination of methods of production and hence of the proportions in which the outputs are being produced to fulfill *social needs* through adaptations of technology.

7. COUNTING OF EQUATIONS III: A MORE SYSTEMATIC PRESENTATION FOR A SPECIAL CASE

The abstract nature and the very generality of Sraffa's approach seems to have prevented theorists from attempting a systematic survey of those links between multiple product industries which lead to an equality of the number of processes and commodities in equilibrium. The preceding examples have shown that joint production of the same commodities on the outputside by means of different methods are not the only constellation which allows the equilibrium condition to be fulfilled. It follows that the transition from single product systems to joint production through the emergence of by-products and the discovery of new, parallel processes which also produce that by-product is only one among several adjustments leading to the dynamic correction of an overdetermination or an underdetermination of prices in consequence of an excess or a deficiency of the number of processes used with respect to the number of commodities produced.

However, because of the traditional emphasis on it, because of its analytical simplicity and because it leads to a straightforward comparison with the neoclassical approach, the case will now be considered in greater detail. The emergence of further products from an "original" single product process leads to shifts of relative (market) prices which allow new links with other existing processes or the introduction of recent inventions to be established such that a new system of prices of production is formed. A by-product will, it is assumed, be needed in definite quantity, either for consumption or as an input to other processes. Then, two main cases may be distinguished:

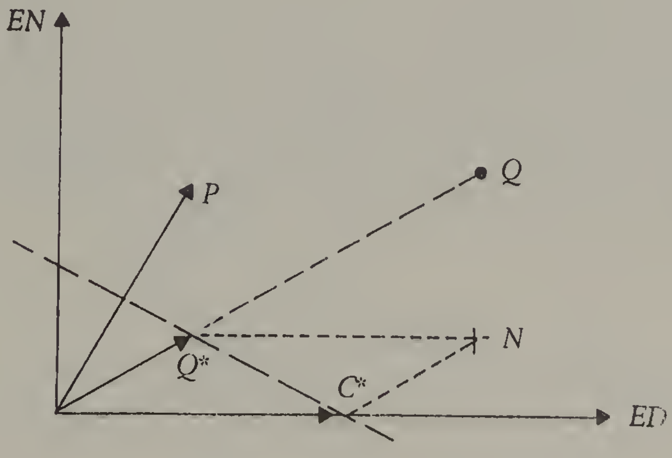
1) Either the original process, run at its original level, provides more than enough of the by-product so that this will not acquire a price and become a commodity.

2) Or the original process, run at its original level, does not provide enough of the by-product. It is then either expanded to the level required by the need for the by-product. In consequence, one should expect the original product to be overproduced and to receive a zero price (case 2a). However, the cheapening of the original product, with its established market, may also lead to the discovery of new uses for it in industrial (case 2b) or in domestic processes of production (case 2c). Or, finally, the original process is not expanded; the excess demand for the by-product will then have to be satisfied by an additional process (case 2d).

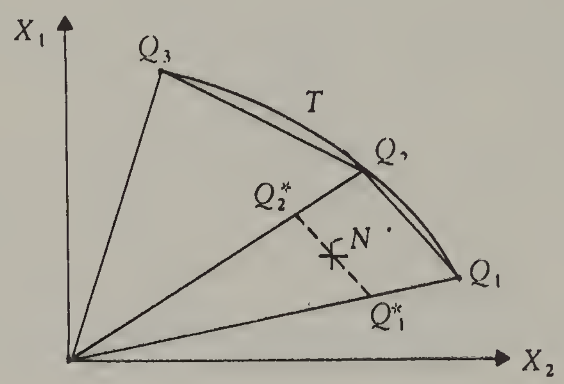
It is clear that analogous distinctions 1a, 1b, 1c, 1d for the first main case are also conceivable, although the existence of overproduction accompanied by a zero price seems most plausible then, because the by-product has never been marketed before. An example of 1a is provided by CO_2 which is a waste product, produced, among other things, jointly with cement, but also absorbed, therefore used as an overproduced input, in the drying of cement.

Insofar as the introduction of a differentiated tariff for electricity produced during the day (*ED*) and the night (*EN*) is older than nuclear power stations, the examples of the preceding sections do not fit into this framework. But insofar as *EN* may be regarded as a by-product of the production of *ED*, of which there is excess production, case 1 can be further illustrated: the case 1b may be identified with $EN \rightarrow ED$ (pumping), 1c is exemplified by $EN \rightarrow H$ (domestic storage heating) and the production of peak-load electricity $C \rightarrow ED$ corresponds to 1d.

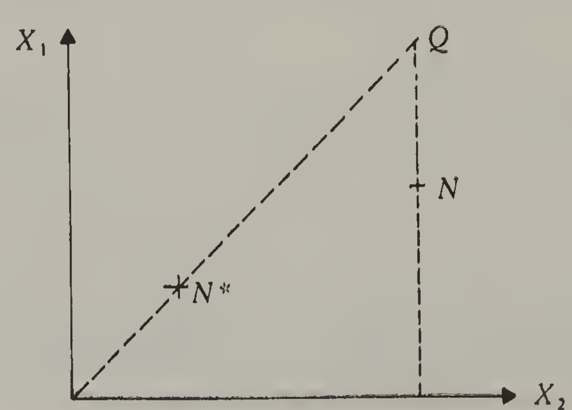
The last transition may be represented graphically in the following diagram showing the outputs of *EN* and *ED* on the axes. *Q* is the production of electricity by means of *NPS* alone such that the needs (point *N*) for *ED* are satisfied; *EN* is then overproduced. An equilibrium corresponding to case 1d is reached, if *NPS* produce only Q^* and coalfired power stations produce the amount C^* of *ED*. If both technologies happen to have the same input costs (including profits and wages), *P* will be the vector of relative prices for *ED* and *EN*.



Similar diagrams are obtained for the other cases. The next diagram shows how the possibility of a continuous substitution on the output side for outputs X_1, X_2 ("transformation curve" *T*) is to be replaced by a finite number of linear segments so that two (Q_2^*, Q_3^*) of the corresponding activities Q_1, Q_2, Q_3 will be used to satisfy given needs N. The transformation curve will by itself consist of linear segments, if discrete linear single product activities have constraints which lead to joint costs (e. g. space in a warehouse where various commodities are stored). The case of continuous substitution was exemplified above by electricity generation with fluidized bed combustion. The relative price is determined by the rate of transformation, but it is not necessary to introduce the rate of substitution derived from utility functions.



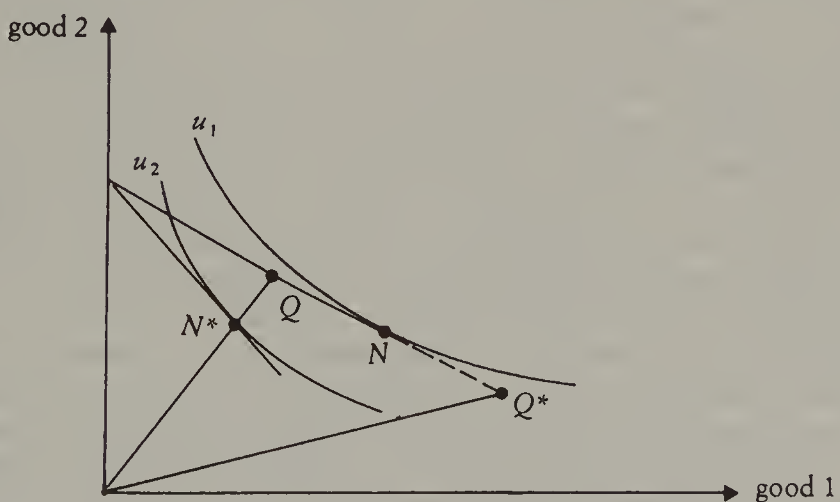
If joint production is rigid (the transformation curve T reduces to a point Q), if no alternative methods for the production of X_1 and/or X_2 are available and if X_1 and X_2 are pure consumption goods (so that there can be no determination of relative prices on the input side), one commodity is necessarily overproduced with respect to needs N and receives a zero price while the production price of the other is determined by cost. The one case resulting in an indeterminacy is the most unlikely and occurs only if, in addition to the preceding conditions, needs N^* happen to be in the same proportion in which the rigid production of the two goods takes place:



34

The theory thus asserts that given social needs can and will in the long run be satisfied through modifications of the system of production. By the same token, the theory denies that discrepancies between the production of original products and by-products on the one hand, and the needs for them on the other, will be matched through adaptations of the needs themselves, i. e. that the needs are so price elastic that changes in *relative prices alone* are sufficient to lead to a permanent adaptation of demand to supply. But such an elasticity of demand, apart from not being necessary according to the approach presented here, is not even plausible on neoclassical assumptions since the original product and the by-product cannot be expected to be substitutes for the same need — if they were, they could for most purposes conveniently be treated in terms of a single commodity. In fact, people are more likely to find new uses for *EN*, to convert *EN* (directly or indirectly) into *ED* or to produce part of *ED* without joint production of *EN* rather than to change their habits and to cook at night only because *EN* is cheaper than *ED*.

To repeat the argument in geometric terms: if needs N are not in line with a given process of production Q, it is not expected that a change in relative prices will cause needs to shift to some point N^* (resulting e. g. from a rise in p_1 with income and substitution effects and depending on indifference curves u_1, u_2) but that other processes (including disposal) will allow to adapt production to demand (as in the diagram, for example, a second process Q^*):



The consumer's sovereignty may thus be posited in the form of rigidly given needs and yet be realised through changes in technology, and this will result in an equilibrium with probability one in a Sraffa system, as was proved in the article "On counting" referred to above and has been shown here in a dynamic context by means of more intuitive methods. (In this paper, the absolute levels of different needs have to be regarded as given because the levels of employment, of wages and profits are given while the theory of steady growth starts from an increase of employment at a given rate, constant returns and from given rates of profits and of wages, hence also from *relative* levels of needs).

The few general hypotheses which have been advanced by economists to explain long term transformations of needs (in particular Engel curves) easily fit into this framework. More importantly, consumption by households may be viewed as the provision of services to fulfill a need by means of domestic processes of production. Insulation as an alternative to heating for the provision of the service "warm house" is a case in point (section 5). It has been observed that the growth of industrial production tends to increase the scope for commodity production at the expense of the domestic provision of services. Preferences for commodities do not seem to be sufficient for the explanation of this change which affects the behaviour of consumers fundamentally.

8. Revival and Development of the Classical Method

In order to analyse the causes which modify the evolution of an economic system over time, the classical system stresses that technology should initially be regarded as given and proposes to consider the various influences leading to its modification only subsequently and one by one. The gradual rise in the consumption of electricity, for instance, leads to decisions concerning the choice of the power stations, this creates after some time an excessive electricity supply at night, and to this various answers can be and are found. The apparent underdetermination of relative prices of outputs is then turned into an overdetermination in that many methods compete for applications and, to the extent that they are actually used, give rise to surplus profits and losses. It is, perhaps, the most remarkable result of this enquiry that the apparent underdetermination of prices which seems to contradict experience, which so much worries the accountants and which leads to the call for a determination of relative prices by means of the neoclassical theory of demand (rate of substitution), may in fact only be a reflection of a lack of perspicacity: in reality, there are often many more processes than commodities in actual use; they — or, rather, the entrepreneurs using them — are rivals trying to establish monopolies and to defend what ought to be temporary rents. The energy producing subsystem is a case in point, for the examples presented in the previous sections all coexist in reality and are expressions of competitive forces in that sector of the economy. If attention is focused directly on a "final equilibrium" in which the technology mix exactly corresponds to demand as derived from preferences, the other forces influencing the chain of events, including what eventually counts as "preferences", will be lost from sight.

As in the examples above, substitution possibilities on the output side play a role at the same time as those between inputs. There is the well known variability of mutton and wool (sheep may be slaughtered earlier or later) but there are also the substitution possibilities on the input side: sweaters can be made from wool and from cotton. Fashion mainly determines the extent to which the substitutability is realized. The possibilities of substitution are very broad where overheads arise because different methods and products are linked only through costs of management and distribution as in a department store. Supply can then be varied to match needs exactly. Rates of transformation are obtained by finding the relevant constraints on production under conditions of normal capacity utilization and afford a rule for splitting up the joint costs.

The likely outcome seems to be a tendency towards overdetermination as I first realized in a discussion of the energy system[10] which quite obviously had to be considered as a group of competing processes with some prices regarded as given and some others as overdetermined. The classical foundation of this analytical approach is the choice of the "socially necessary technique". As we shall see, even the presence of "too many" processes in all sectors of the economy does not represent a chaotic state of overdetermination, since it may be assumed that the socially necessary technique has already been determined in all sectors of the economy except the one under consideration so that, in the case of single product systems, input prices in any given industry may be regarded as already determined. It can then be discussed which method (or which combination of methods) is socially necessary and determines output prices in the sector under consideration, while other methods yield rents.

Not sufficient attention has been paid so far to the question of how this procedure is to be made more rigorous in basic Sraffa systems (where the output price reacts back on input prices), to joint production (where the classification of "industries" is not straightforward) and to conditions where there remain "pockets of underdetermination".

According to one approach, the decisions about the methods of production to be regarded as "socially necessary" (e. g. solar versus nuclear energy) have to be taken in the light of estimates about the potential productivity growth of each, about the supply of raw materials, about political developments and the evolution of social values ("preferences"). Ricardo thus treated as socially necessary that technique which allowed to satisfy *total* demand in the long run in the cheapest way, i. e. the technique employed on the marginal land in agriculture and the most productive technique in manufacturing industry.

[10] B. Schefold, "Energy and Economic Theory", *Zeitschrift für Wirtschafts- und Sozialwissenschaften*, 1977, pp. 227-249.

Alternatively, and for more short run considerations, an average of existing methods in each sector of the economy has been defined as the socially necessary technique. This was the approach suggested by Marx in vol. III of "*Das Kapital*" in the determination of what he called "market value", while his concept of "socially necessary labour" in vol. I is, apart from the political and historical element, closer to Ricardo. The same principle of "averages" is also used in the construction of modern input-output tables where the coefficients reflect the average productivity in any industry.

In the future, it may become possible to extend Sraffa's theory of joint production further by developing new practical rules for aggregation and by rearranging the statistical data in order to construct *square* joint production input *and output* tables.

The socially necessary technique of classical economics is thus found by applied economists either by means of "technology assessments" or by means of "aggregation". The purpose is in both cases to abstract from short run disturbances (cf. the "market prices" of the classics). Both methods may have to be combined in order to deal with all possible situations. The "pockets of underdeterminacy" are eliminated if technology assessment yields estimates about limits to the range in which an underdetermined price may move (e. g. the future price of gasoline sets a limit to the possible variation of the price of liquefied coal as a future by-product of high temperature nuclear reactors). If this is not feasible, the product is likely to be new and therefore non-basic so that it is eliminated in the formation of the basic system.

The familiar construction of the basic system is therefore rendered more complicated only insofar as there will be overdetermination. A theoretically rigorous solution is then available: if competition prevails, that combination of processes will be chosen which maximizes the rate of profit, given the real wage[11]. However, in concrete circumstances the choice between "technology assessment" (selection of a method which is likely to dominate and which does not necessarily minimize costs) and "aggregation" (formation of averages) is to some extent a matter of judgement and of purpose: technology assessment is more appropriate for prediction and policy-making while aggregation serves the analysis of inter-industry relationships in the present. In practice, the availability of data and the access to information about industrial strategies will be decisive factors. Finally, where surplus profits have been consolidated into rents, the traditional method should be followed and the "marginal" process determines prices.

But, whatever method is chosen in the formation of the basic system from "socially necessary techniques", the classical method of regarding distribution, employment and the composition of output as given is clearly an essen-

[11] B. SCHEFOLD, "Von Neumann and Sraffa: Mathematical Equivalence and Conceptual Difference", *The Economic Journal*, March 1980, vol. 90, pp. 140-156.

tial element in the derivation of the equations which determine production prices; it is difficult to see how these procedures could be used in a neoclassical general equilibrium approach although they are, apart from the element of joint production, standard in input-output analysis (with rates of profit differing between sectors).

Once production prices have, empirically or conceptually, been derived at some level of aggregation, a disaggregation for a sector under particular consideration is again feasible in order to discuss the underdetermination or overdetermination of certain prices (other prices being given), with competing processes and the potential introduction of by-products, as we have done with examples taken mainly from the energy sector in sections 5, 6 and 7 above.

The classical economists themselves do not seem to have been aware of the possibility that production prices might have to be determined simultaneously in a group of joint production processes. They treated joint production as a special extension of single product industries by implicitly distinguishing between the main product (whose price was equal to cost of production) and a subsidiary product which would fetch as much as a close substitute. This is why Adam Smith argued that fur was cheap in a country where fur was used for clothing and where the main diet was meat[12], and why Marx called joint products the "excretions of production" and seemed to think that the problem of joint production was mainly that of capitalists trying to sell waste products to make some additional profit[13].

Natural processes — if we may use that expression — always feature joint production in that any transformation in an ecological system can be regarded to fulfill several functions, but the purposive act of human production is typically directed at the creation of *one* product. This, and the difficulty of accounting for costs and prices in the case of joint production, lead to the prevalence of single product systems and the emphasis on single production by the political economists in the period of classical analysis. I interpret the paper by Kurz[14] — which had been sent to me when this paper had already gone to the Journal — as yielding further evidence on this point. The classics did discuss disposal activities but were unable to provide a complete solution to the determination of relative values of joint products — hence Jevons's famous scorn for J. S. Mill who had invoked "supply and demand" in this context. But it seems also plausible that the increasing importance of chemical industry in the last third of the 19th century led to a greater awareness of the problem. At any

[12] H. D. Kurz, "Joint Production and the Influence of Demand on Relative Prices: A. Smith, J. St. Mill and Marshall", *mimeo*, 1980.

[13] K. Marx, *Das Kapital*, Band III (1894), Berlin, Dietz Verlag 1969, pp. 110-113.

[14] H. D. Kurz, "Joint Production in the History of Economic Thought", *mimeo* (60 pp.), 1984.

rate, I should like to advance the hypothesis for further historical investigation that the processes introduced in the manufacturing sector during the first industrial revolution lead less often to immediate joint production of commodities than the more traditional methods preceding and the more advanced methods following it.

Today, the classical theory of value can be extended quite naturally to multiple product industries, if Sraffa's suggestion "to count the equations" is taken up, as we have seen in this paper. Sraffa had discovered that the underdeterminacy of one process with two joint products leaves "room for a second, parallel process". Now we have seen that there are *incentives* actually to bring the number of processes used and commodities produced into equality. The point, however, is to transcend the formal analysis of the long period equilibrium and to analyse the movement of the economy by taking the long period position only as the frame of reference without identifying it with the actual state of the economy. Since the properties of joint production systems in equilibrium have largely been clarified, the task is to analyse movements of capital between industries which equalize the rates of profit. The obstacles to the tendency of an equalization of the rates of profit are diverse, but in the course of the evolution of the economic system institutions are being developed which support that tendency, such as the capitalization of rents and the *ex post* reevaluation of the means of production in capital markets. In a sense, the phenomenon which we have here classed under the heading of "counting of equations" represents a specific form which this tendency assumes in the case of joint production.

However, it is not possible to analyse the changes of prices, distribution and technology all simultaneously. One can, given distribution and the methods of production, analyse the formation of a general rate of profit with the transfer of capital between industries but such was not our purpose here. We have analysed the competition between methods of production at a given general rate of profit and at given levels of demand in terms of surplus profits which may be turned into rents in the case of an overdetermination and which are due to the profit contribution of excess commodities in the case of an underdetermination of prices. The basis of our analytical procedure therefore was provided by the classical methodological separation between the theories of distribution, value and output. It is this which had allowed us to identify the classical element in modern econometric models and which allowed us to reconcile the apparent contradiction between the conflicting interpretations of joint production in mathematical economics on the one hand and business administration on the other.

Institut für Markt und Plan,
Johann Wolfgang Goethe- Universität, Frankfurt am Main.

[8]

The Economic Journal, **87** (*June* 1977), 324–328
Printed in Great Britain

BASICS, NON-BASICS AND JOINT PRODUCTION*

The distinction between basic and non-basic commodities plays a central role in the analysis presented by Sraffa in his *Production of Commodities by Means of Commodities* (1960). It may therefore be of interest to examine rather closely the general formulation of the distinction between basics and non-basics, which Sraffa presents in the course of his discussion of joint production.

While Sraffa represents the inputs and outputs for a given productive process by the corresponding *rows* of the input and output matrices, the more common practice of representing them by the *columns* will be adopted here. Subject to this slight change, Sraffa's general formulation of the distinction between basic and non-basic commodities is as follows (1960, § 60):

> In a system of k productive processes and k commodities (no matter whether produced singly or jointly) we say that a commodity or more generally a group of n linked commodities (where n must be smaller than k and may be equal to 1) are non-basic if of the k [columns] (formed by the $2n$ quantities in which they appear in each process) not more than n [columns] are independent, the others being linear combinations of these.

PASINETTI'S DIRECT AND INDIRECT CAPITAL MATRIX

In an article published in *Metroeconomica* (1973), Pasinetti introduced the concept of a "direct and indirect capital matrix", the jth column of which is the vector of means of production required, directly or indirectly, for the production of one unit of *net* output of commodity j. If the inputs and outputs for a given productive process are represented by the corresponding columns of matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then it is easily shown (see §§4, 13) that the direct and indirect capital matrix, $\mathbf{H}$, is given by

$$\mathbf{H} \equiv \mathbf{A}(\mathbf{B}-\mathbf{A})^{-1}. \tag{1}$$

BASICS, NON-BASICS AND THE H MATRIX

In a single product system commodities can be so numbered that the $\mathbf{A}$ matrix takes the form

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ 0 & \mathbf{A}_4 \end{bmatrix},$$

where matrix $\mathbf{A}_1$ refers to basics, and $\mathbf{B} \equiv \mathbf{I}$. Then the $\mathbf{H}$ matrix, since

$$\mathbf{H} \equiv \mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} \equiv \mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\dots$$

here, takes the form

$$\mathbf{H} \equiv \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ 0 & \mathbf{H}_4 \end{bmatrix},$$

* I should like to thank L. L. Pasinetti for encouragement and helpful discussion and a referee for useful criticisms of an earlier version of this note.

where $\mathbf{H}_1 \equiv \mathbf{A}_1(\mathbf{I}-\mathbf{A}_1)^{-1}$, $\mathbf{H}_4 \equiv \mathbf{A}_4(\mathbf{I}-\mathbf{A}_4)^{-1}$. While no comparable re-ordering can be done for the $\mathbf{A}$ and $\mathbf{B}$ matrices of a joint production system, since they are commodity/process matrices and not commodity/commodity matrices, it is natural to ask whether the corresponding $\mathbf{H}$ matrix, which *is* a commodity/commodity matrix, might not exhibit the classification of commodities into basics and non-basics in just the same way as occurs with single products.

Following the argument of Manara (1968, §6), re-order $\mathbf{A}$ and $\mathbf{B}$ so that the first j rows refer to basics and the last m rows to non-basics and so that the last m columns are linearly independent in their last m rows.[1] Now writing

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \mathbf{B} \equiv \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix},$$

Sraffa's general definition of non-basics implies that there exists a matrix $\mathbf{T}$ such that $\mathbf{A}_3 \equiv \mathbf{A}_4\mathbf{T}$ and $\mathbf{B}_3 \equiv \mathbf{B}_4\mathbf{T}$. Defining

$$\mathbf{M} \equiv \begin{bmatrix} \mathbf{I}_j & \mathbf{o} \\ -\mathbf{T} & \mathbf{I}_m \end{bmatrix},$$

it follows that

$$\mathbf{AM} \equiv \begin{bmatrix} \mathbf{A}_1 - \mathbf{A}_2\mathbf{T} & \mathbf{A}_2 \\ \mathbf{o} & \mathbf{A}_4 \end{bmatrix} \quad \text{and} \quad \mathbf{BM} \equiv \begin{bmatrix} \mathbf{B}_1 - \mathbf{B}_2\mathbf{T} & \mathbf{B}_2 \\ \mathbf{o} & \mathbf{B}_4 \end{bmatrix}.$$

Now, since $\mathbf{H}(\mathbf{B}-\mathbf{A}) \equiv \mathbf{A}$, we have $\mathbf{H}(\mathbf{BM}-\mathbf{AM}) \equiv \mathbf{AM}$, which leads easily to

$$\mathbf{H} \equiv \left[\begin{array}{c:c} (\mathbf{A}_1 - \mathbf{A}_2\mathbf{T})(\mathbf{C}_1 - \mathbf{C}_2\mathbf{T})^{-1} & [\mathbf{A}_2 - (\mathbf{A}_1 - \mathbf{A}_2\mathbf{T})(\mathbf{C}_1 - \mathbf{C}_2\mathbf{T})^{-1}\mathbf{C}_2]\,\mathbf{C}_4^{-1} \\ \hdashline \mathbf{o} & \mathbf{A}_4\mathbf{C}_4^{-1} \end{array}\right], \qquad (2)$$

where $\mathbf{C}_i \equiv (\mathbf{B}_i - \mathbf{A}_i)$.

It will be seen immediately from (2) that Sraffa's general definition of basics and non-basics does indeed imply that, after suitable relabelling of commodities, the $\mathbf{H}$ matrix has an $(m \times j)$ block of zeros in the lower left-hand submatrix ($\mathbf{H}_3 \equiv \mathbf{o}$). In this respect then, the similarity with the $\mathbf{H}$ matrix for the single product case is very strong.

The similarity is also strong with respect to the lower right-hand submatrix (the matrix $\mathbf{H}_4$), since $\mathbf{H}_4 \equiv \mathbf{A}_4\mathbf{C}_4^{-1}$ in both the single products and the joint products case. On the other hand, while $\mathbf{H}_1 \equiv \mathbf{A}_1\mathbf{C}_1^{-1}$ in the single products case, this is *not* so, in general,[2] with joint production. In the general joint production case the submatrix $\mathbf{H}_1$ depends on $\mathbf{T}$ and it therefore depends on the conditions of production of the non-basic commodities.

By a suitable choice of units, let the *net* output of each commodity be equal to unity. Then the vector of means of production available at the beginning of the production period, in the economy as a whole, is simply given by the sum of the

[1] Read strictly, Sraffa's general definition, quoted above, implies only that "not more" than m of the last columns are linearly independent in their last m rows and, indeed, in the footnote to his general definition Sraffa says explicitly that "less than" m columns could be linearly independent. There is good reason to think, however, that Manara is right to insist on the presence of *exactly m* such columns; see the Appendix to the present note.

[2] It is easy to see that $(\mathbf{A}_1 - \mathbf{A}_2\mathbf{T})(\mathbf{C}_1 - \mathbf{C}_2\mathbf{T})^{-1} = (\mathbf{A}_1\mathbf{C}_1^{-1})$ if and only if $(\mathbf{A}_2\mathbf{T}) = (\mathbf{A}_1\mathbf{C}_1^{-1})(\mathbf{C}_2\mathbf{T})$: a sufficient but not necessary condition is thus obviously that $\mathbf{T} = \mathbf{o}$, i.e. $\mathbf{A}_3 = \mathbf{B}_3 = \mathbf{o}$.

columns of **H**. More significantly, the individual columns of **H** provide a *conceptual* division of that total vector of means of production into $(j+m)$ separate vectors of means of production corresponding to the net outputs of the $(j+m)$ individual commodities. In other words, column k of **H** shows the means of production used in a hypothetical "subsystem", the net output from which consists simply of one unit of commodity k. Thus the fact that $\mathbf{H}_3$ is null shows that no non-basic commodity enters into the means of production used in the (hypothetical) "subsystem" for any basic commodity, whereas the converse is not true ($\mathbf{H}_2$ is not null). Allowing for joint-production does, however, make a difference to the interpretation of the columns of **H**, since it is now possible (though not necessary) that some elements of **H** will be *negative*, which cannot occur in single-product systems. (It is the presence of such negative elements which can give rise to negative components of the Standard Commodity (Sraffa, 1960, §56) and to a *positive* relation between the real wage and the rate of profits for given methods of production (*ibid.*, §72).[1])

It seems natural at this point to suggest that one might now *define* basics as those commodities which appear in the $\mathbf{H}_1$ submatrix when **H** has been reduced as far as possible by suitable relabelling of commodities. This definition both reduces to the more familiar definition in terms of the submatrix $\mathbf{A}_1$ in the single products case and surely has far greater intuitive appeal than Sraffa's formulation of the general definition in terms of linear dependence and independence.

PRICES AND THE RATE OF PROFITS

Consider first the case in which wages are zero, so that the row vector of prices, $\mathbf{p}'$, and the rate of profits, R, must satisfy:

$$\left.\begin{aligned} &(1+R)\,\mathbf{p}'\mathbf{A} = \mathbf{p}'\mathbf{B} \\ \text{and hence}\quad &R\mathbf{p}'\mathbf{A} = \mathbf{p}'(\mathbf{B}-\mathbf{A}) \\ \text{and}\quad &R\mathbf{p}'\mathbf{H} = \mathbf{p}' \end{aligned}\right\} \tag{3}$$

from (1). Partitioning the price vector in two, referring to basics as b and non-basics as nb respectively, and partitioning **H** as in (2), (3) may be rewritten as

$$R[\mathbf{p}_b' \quad \mathbf{p}_{nb}'] \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{o} & \mathbf{H}_4 \end{bmatrix} = [\mathbf{p}_b' \quad \mathbf{p}_{nb}']. \tag{4}$$

In particular,

$$\begin{aligned} R\mathbf{p}_b'\mathbf{H}_1 &= \mathbf{p}_b' \\ &= R\mathbf{p}_b'(\mathbf{A}_1-\mathbf{A}_2\mathbf{T})\,(\mathbf{C}_1-\mathbf{C}_2\mathbf{T})^{-1}. \end{aligned} \tag{5}$$

Setting aside, on familiar grounds, the solution $\mathbf{p}_b' = \mathbf{o}'$, it is seen at once from (5) that both R and $\mathbf{p}_b'$ depend on the matrix **T**.

Consider, then, two different economies which produce the same commodities, which are such that the same commodities are basics and in which, indeed, the inputs and outputs of basics, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$ are identical. The two economies

[1] Of course, such a positive relation cannot obtain if the real wage is measured in terms of the Standard Commodity.

differ, however, in that the appropriate **T** matrix differs as between them. It follows, from (5), that, in general, R and $\mathbf{p}_b'$ will differ as between the two economies. Remembering that the matrix **T** relates to the conditions of production of non-basics, it is not entirely clear how one is to interpret Sraffa's statement (1960, §65) that, even in joint-production systems, "basics have an essential part in the determination of prices and the rate of profits, while non-basics have none". While it is perfectly true that Sraffa's basic *equations* (*ibid.* §62) suffice to determine basic prices and the rate of profits, one cannot deduce that only basic *commodities* play a role in that determination.

It does not follow, however, that the distinction between basics and non-basics has become an empty one. It is still the case that, *given* the conditions of production (of both basics and non-basics), the relation between the prices of basics and the rate of profits can be considered independently of the relation between the prices of non-basics and the rate of profits, while the converse is not true. Thus (4) can be generalised as follows: let wages be positive and paid at the end of the production cycle; let r be the rate of profits; let $\mathbf{p}'$ be a row vector of labour-commanded prices and $\mathbf{v}'$ a row vector of direct and indirect labour use (Marxian values). It can then be shown (Pasinetti, 1973, §§12, 13) that, in partitioned form, (4) is generalised to[1]

$$[\mathbf{p}_b' \quad \mathbf{p}_{nb}'] = [\mathbf{v}_b' \quad \mathbf{v}_{nb}'] + r[\mathbf{p}_b' \quad \mathbf{p}_{nb}'] \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{o} & \mathbf{H}_4 \end{bmatrix}. \tag{6}$$

It will be seen from (6) that

$$\mathbf{p}_b' = \mathbf{v}_b'(\mathbf{I} - r\mathbf{H}_1)^{-1}, \tag{7}$$

$$\mathbf{p}_{nb}' = (\mathbf{v}_{nb}' + r\mathbf{p}_b'\mathbf{H}_2)(\mathbf{I} - r\mathbf{H}_4)^{-1}. \tag{8}$$

Thus the functional relation between $\mathbf{p}_b'$ and r can be examined independently of that between $\mathbf{p}_{nb}'$ and r (equation (7)) but the latter depends on the former (equation (8)): it is not, of course, implied that the relation between $\mathbf{p}_b'$ and r is independent of the conditions of production of non-basics, since both $\mathbf{v}_b'$ and $\mathbf{H}_1$ depend on **T**.

CONCLUSION

It has been suggested that an intuitively appealing definition of a basic commodity can be retained, in the general joint products case, provided that it is given in terms of the Pasinetti **H** matrix. After appropriate relabelling, the **H** matrix will exhibit the basic commodities as the irreducible square upper left-hand submatrix, while the lower left and right-hand submatrices, referring to non-basics, will be $\mathbf{H}_3 \equiv \mathbf{o}$ and $\mathbf{H}_4 \equiv \mathbf{A}_4\mathbf{C}_4^{-1}$, just as in the single products case.

[1] Relation (6) can, of course, be written in unpartitioned form as

$$\mathbf{p}' = \mathbf{v}' + r\mathbf{p}'\mathbf{H}.$$

It is interesting to note that this relation implies the existence of "families" of joint production systems, each member of a family having a different **A**, **B**, $\mathbf{a}'$ ($\mathbf{a}'$ being the row vector of direct labour inputs) but the *same* **H**, $\mathbf{v}'$ and hence the *same* economic properties. System j will belong to the same family as system 1 if and only if $\mathbf{A}_j\mathbf{B}_j^{-1} = \mathbf{A}_1\mathbf{B}_1^{-1}$ and $\mathbf{a}_j'\mathbf{B}_j^{-1} = \mathbf{a}_1'\mathbf{B}_1^{-1}$. (Cf. Schefold (1971), theorem 3.1; Schefold also makes the point (p. 11) that the prices of basics have a logical priority over those of non-basics.)

(It must be noted, however, that both $\mathbf{H}_1$ and $\mathbf{H}_2$ depend on the matrix $\mathbf{T}$ and hence on the production conditions of non-basics.) It has also been shown that, in general, non-basic commodities do have a role in the determination of prices and the rate of profits but that the prices of basics still have a logical priority over the prices of non-basics.

IAN STEEDMAN

University of Manchester

Date of receipt of final typescript: November 1976

REFERENCES

Manara, C. F. (1968). "Il Modello di Piero Sraffa per la Produzione Congiunta di Merci a Mezzo di Merci." *L'Industria*, pp. 3–18.

Pasinetti, L. L. (1973). "The Notion of Vertical Integration in Economic Analysis." *Metroeconomica*, pp. 1–29.

Schefold, B. (1971). *Mr Sraffa on Joint Production*. Basel: privately published.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.

APPENDIX

The purpose of this appendix is to indicate why Manara was correct to postulate the existence of a number of linearly independent columns *exactly equal* to the number of non-basics, rather than allowing the number of such columns to be merely less than or equal to the number of non-basics, as is suggested by Sraffa.

Using the notation introduced above, consider the simple case in which $\mathbf{A}_3 = \mathbf{B}_3 = \mathbf{T} = 0$. Taking a feasible rate of profits, $\bar{r}$, as given, the labour-commanded prices must satisfy the relations:

$$(1+\bar{r})\,\mathbf{p}_b'\mathbf{A}_1 + \mathbf{a}_1' = \mathbf{p}_b'\mathbf{B}_1, \tag{i}$$

$$(1+\bar{r})\,(\mathbf{p}_b'\mathbf{A}_2 + \mathbf{p}_{nb}'\mathbf{A}_4) + \mathbf{a}_2' = \mathbf{p}_b'\mathbf{B}_2 + \mathbf{p}_{nb}'\mathbf{B}_4, \tag{ii}$$

where $\mathbf{a}_1'$ refers to the first j processes and $\mathbf{a}_2'$ to the last m. The prices p_b' are determined by (i) as

$$\mathbf{p}_b' = \mathbf{a}_1'[\mathbf{B}_1 - (1+\bar{r})\,\mathbf{A}_1]^{-1}$$

and (ii) then yields

$$\mathbf{p}_{nb}'[\mathbf{B}_4 - (1+\bar{r})\,\mathbf{A}_4] = \mathbf{a}_2' - \mathbf{p}_b'[\mathbf{B}_2 - (1+\bar{r})\,\mathbf{A}_2]. \tag{iii}$$

Now the matrix $[\mathbf{B}_4 - (1-\bar{r})\mathbf{A}_4]$ in (iii) is, of course, a square $(m \times m)$ matrix. If, following Manara, we take that matrix to contain m linearly independent columns, then it can be inverted to solve (iii) for the prices $\mathbf{p}_{nb}'$. On the other hand if, following Sraffa, we allow the possibility that the rank of $[\mathbf{B}_4 - (1+\bar{r})\,\mathbf{A}_4]$ might be *less than m*, then, if that possibility should be realised, either (iii) will yield no solution at all for $\mathbf{p}_{nb}'$ or, if solutions do exist, they will not be unique.

[9]

The Economic Journal, **92** (*June* 1982), 377–385
Printed in Great Britain

JOINT PRODUCTION AND THE WAGE–RENT FRONTIER*

Ian Steedman

It is well known that in any constant-returns-to-scale production model with single-product processes, homogeneous labour and homogeneous land, the real wage rate is inversely related to the real rent rate, for any sensible measure of real wages and rents. It will be shown below, by means of a simple, two-commodity example, that the same cannot be said *a priori* when joint production is allowed. Some further consequences of this fact will be set out and the results will then be generalised. Since the theory of single-product processes is predominant in some parts of the literature, whilst joint-product processes are predominant in many parts of the real world, it is important to show *explicitly* that some familiar theorems of single-products theory do not hold good, without modification, in the joint-products context.

I. PRODUCTION METHODS AND THE PRODUCTION POSSIBILITY FRONTIER

Consider an economy in which the only available, constant-returns production processes are those shown in Table 1. It will be noted that, for the sake of simplicity, no process is assumed to use produced means of production and that, while processes 1, 3 and 4 are single-product processes, process 2 produces both commodities.

If the fixed supplies of homogeneous labour and homogeneous land are 8 units and 12 units, respectively, the production possibility frontier is *ABCDE* in Fig. 1. At *A*, only P_1 is used; along *AB*, both P_1 and P_2 are used; at *B*, P_2 alone is operated; along *BC* and at *C*, both P_2 and P_3 are in use; along *CD*, P_2, P_3 and P_4 are all used; at *D*, along *DE* and at *E*, P_2 and P_4 are employed. From *A* to *C*, excluding *C* itself, there is unused land, but from *C* to *E* both labour and land are fully utilised. The section *DE* is vertical because it represents production at *D* combined with 'free disposal' of varying amounts of commodity 2.

It will be seen that the absolute slopes of *AB*, *BC* and *CD* are $\frac{1}{4}$, 1 and 3, respectively, so that it is clear which possible output will maximise the value of national income for any $p \equiv (p_1/p_2)$, where p_i is the price of commodity i.

II. A WAGE–RENT FRONTIER

We now consider how the real wage rate and the real rent rate *measured in terms of commodity 2*, w_2 and W_2 respectively, will vary as the commodity price

* I should like to thank J. M. Currie, J. Eatwell, M. C. Kemp, H.-D. Kurz, J. S. Metcalfe, C. Montet, N. Salvadori and anonymous referees for encouragement and helpful suggestions.

Table I

Process	Labour		Land		Commodity I		Commodity 2
P_1	4	+	5	→	0	+	5
P_2	1	+	1	→	1	+	1
P_3	1	+	3	→	2	+	0
P_4	5	+	9	→	8	+	0

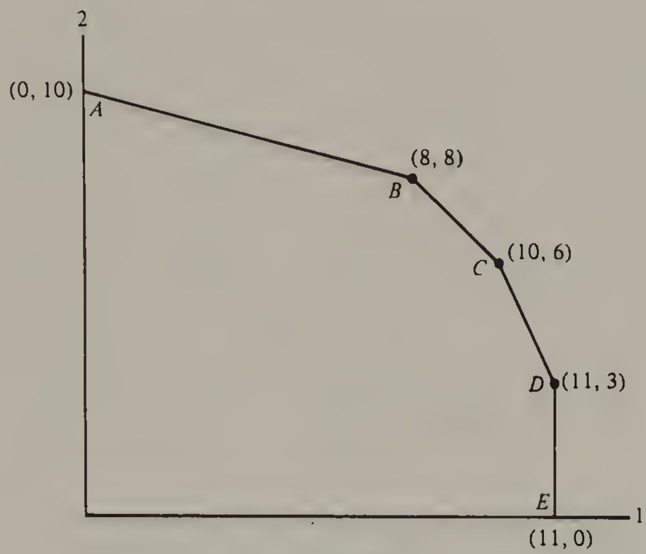


Fig. 1

ratio, p, increases from zero. For $0 \leqslant p \leqslant \frac{1}{4}$, only P_1 will be used and W_2 will be zero, because land is underutilised: it is thus clear from the first row of Table 1 that $4w_2 = 5$. For $\frac{1}{4} < p < 1$, only P_2 will be used and W_2 will still be zero: from the second row of Table 1, $w_2 = (p+1)$. With $1 < p < 3$, both P_2 and P_3 will be used and W_2 will now be positive: from the second and third rows of Table 1, $(w_2 + W_2) = (p+1)$ *and* $(w_2 + 3W_2) = 2p$. Hence $2w_2 = (p+3)$ and $2W_2 = (p-1)$. Proceeding in this way, we may complete Table 2.

It will be noted that all the five entries in Table 2 which vary with p do so positively. It follows at once that Table 2 sets out (implicitly) an *upward sloping* real wage–real rent frontier; it is shown explicitly in Fig. 2, where the arrows show the direction of movement as p increases. (Note that the third 'branch' extends indefinitely and that the branch labelled 'b' corresponds to corner 'B' in Fig. 1, and so on.)

However trivial the above example may be thought to be, it naturally suffices to show that, with joint production present, one *cannot* assert that real wages and real rents must be inversely related.

Table 2

	$0 \leqslant p \leqslant \frac{1}{4}$	$\frac{1}{4} \leqslant p \leqslant 1$	$1 \leqslant p \leqslant 3$	$3 \leqslant p$
$4w_2$	5	$4p+4$	$2p+6$	$p+9$
$4W_2$	0	0	$2p-2$	$3p-5$

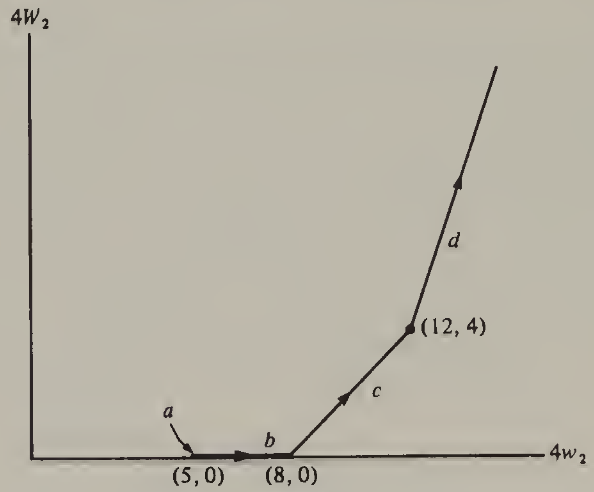


Fig. 2

III. OTHER WAGE–RENT FRONTIERS

It has not been said that, even in the present example, the real wage–real rent frontier necessarily slopes upwards: by contrast with the single-products case, when there is joint production the very direction of movement along such a frontier can depend on the standard in which real wages and rents are measured. That it does so depend in our example is shown in Fig. 3; in (*a*) real wages and rents are measured in terms of commodity 1, while in (*b*) they are measured in terms of the standard $(p_1+6p_2) = 1$. In each case, the arrows again show the direction of movement as p increases. The W_1/w_1 frontier in Fig. 3(*a*) is, of course, more similar than is the W_2/w_2 frontier to that for a single-products case: but note that even the W_1/w_1 frontier does *not* reach the W_1 axis as p increases without limit. In Fig. 3(*b*), by contrast with Figs. 2 and 3(*a*), the real wage is alternately falling, rising and falling again; as p increases monotonically from zero to infinity, $840w$ *falls* from 175 to 168, then *rises* to 280 and then *falls* to 210. Note that, as a consequence, there are two values of the real rent compatible with each real wage rate such that $3 \leqslant 12w < 4$: with joint production, the real wage/real rent relation may not be single-valued. As in Fig. 3(*a*), the frontier does *not* reach the real rent axis as p increases without limit.

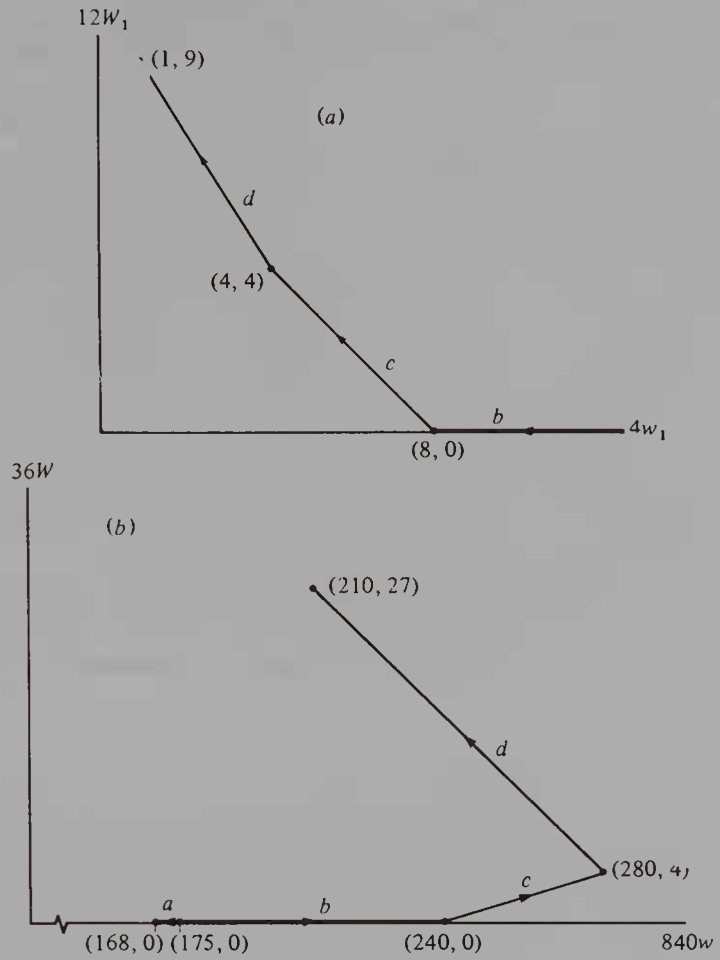


Fig. 3

IV. ALTERNATIVE CONTEXTS

We now return to Fig. 2 and to measurement of real wages and rents in terms of commodity 2. Each upward-sloping portion of the frontier in Fig. 2 corresponds to production somewhere on *CDE* in Fig. 1 and thus to the use of process P_2 together with P_3 and/or P_4. Since it is clear from Table 1 that, when such combinations of processes are employed, it is impossible to produce exactly an output consisting solely of commodity 2, it might be thought to follow that the 'unusual' behaviour of W_2 and w_2 is thereby deprived of any interest or significance. Such a thought would be unjustified.[1]

Thus suppose first that our economy is a small, open economy, facing given terms of trade $p > 1$, and that, in this economy, consumption consists of commodity 2 alone. It will now be true both that production will take place on *CDE* and that commodity 2 is a perfectly sensible standard in which to

[1] Unjustified, that is, even on the part of one who supposes that 'real' measurement of the wage-rate and rent-rate, in terms of some standard of value, is of significance only if such a measurement can be given a 'welfare' interpretation.

measure wages and rents. The upward-sloping part of Fig. 2 is now a significant wage–rent frontier to consider as the given terms of trade vary parametrically.

It is not suggested, however, that the small, open economy interpretation is the only way of giving significance to the measurement of wages and rents in terms of commodity 2, while production is on *CDE*.[1] Thus we could continue to suppose that commodity 2 is the only commodity consumed by the recipients of wages and rents but introduce a government which taxes wages and rents and spends its revenue wholly or partially on commodity 1. Or we could introduce produced means of production – consisting solely of commodity 1 for simplicity – and suppose that savings, net investment and growth takes place. ('Land' and 'rent' would more appropriately be called 'skilled labour' and 'the skilled wage rate' in this case, of course.) Or we could introduce both produced means of production and a class of capitalists, who receive interest and spend it wholly or partially on commodity 1. Again, as will be seen below, the presence of sales taxes can give significance to the measurement of wages and rents in terms of commodity 2.

There is thus no reason to reject as uninteresting the measurement of wages and rents in terms of a commodity bundle which cannot be produced by the production methods which yield an upward-sloping wage–rent frontier.

V. SOME IMPLICATIONS

Before generalising the argument in the next section, we may consider some further implications of the presence of joint production, in the context of our simple example.[2] It will have been noticed from Figs 2, 3 (*a*) and 3 (*b*) that, when rents are positive, p is positively related to the rent–wage *ratio*: in interpreting the following comparative statics propositions, one may therefore be tempted to identify commodity 1 as the land-intensive commodity and commodity 2 as the labour-intensive commodity.

(1) *A Shift in Demand*

Suppose that all consumers have the same homothetic preference map and consider the effect of a shift of preferences, in logical time, in favour of commodity 1. In a closed economy, the equilibrium value of p will rise and hence that of (W/w) will also rise. As has been seen, however, the value of the real wage rate will not fall in terms of all standards of value. Thus, in terms of some value standards, a shift in demand towards commodity 1, the *land*-intensive commodity, will cause an increase in the real return to *labour*. The point made here is not, of course, that a 'perverse' relationship between commodity demand and factor returns is to be expected in the presence of

[1] That interpretation does, nevertheless, deserve to be given greatest emphasis; first, because of the importance of the 2×2 model, the Rybczynski theorem, the Stolper–Samuelson theorem, etc., in trade theory and, secondly, because by ignoring produced means of production (see below) one makes clear that the possibility of an upward-sloping wage–rent frontier stems from joint production alone, having nothing to do with capital theory phenomena.

[2] Throughout this section we return to the simple case in which there are no capitalists and, except when a tariff is considered, no government. The economy will be taken to be closed, again except when a tariff is introduced into the argument.

joint production but only that, with joint production, important relationships may hold good in terms of some value standards but not in terms of others.[1]

(2) *An Increase in Labour Supply*

Ceteris paribus, if the labour supply is increased from 8 to 11 units in our numerical example, point C in Fig. 1 will shift from the point (10, 6) to ($11\frac{1}{2}$, $10\frac{1}{2}$); the 'new' point C' has a higher ratio of output of commodity 2 to that of commodity 1. If the common homothetic preference map is such that the 'old' equilibrium, in a closed economy, lies at C and the 'new' one at C' (as is perfectly possible), it follows that the 'new' equilibrium will have a higher p and hence a higher ratio (W/w). Yet the 'new' real wage rate will be higher in terms of some standards of value: in some standards, the *higher* labour supply will be associated with the *higher* real wage rate. Once again, the point being made is not, of course, that joint production leads one to expect a 'perverse' relation between factor supplies and factor returns but simply that the very nature of that relation may depend on how the returns are measured (which is not the case in single-product systems).

(3) *The Rybczynski Theorem*

That the Rybczynski theorem cannot be extended in a direct way to the case of joint production can be seen at once by developing the example of the previous paragraph. Just as point C, in Fig. 1, moves from (10, 6) to ($11\frac{1}{2}$, $10\frac{1}{2}$), when the labour supply increases from 8 to 11 units, so point D moves from (11, 3) to ($11\frac{3}{4}$, $9\frac{3}{4}$). Both the 'old' and the 'new' CD have an absolute slope of 3, but the 'new' CD lies unambiguously above and to the right of the 'old' one. Hence at a fixed commodity price ratio, $p = 3$, the *ceteris paribus* increase in the labour supply leads to an increase in *both* outputs. Whichever of the two commodities is taken to be the land-intensive one, its output has *not* been decreased by the increase in the supply of labour, at a constant price ratio.

(4) *The Stolper–Samuelson Theorem*

As might be expected in the light of the previous paragraph, the Stolper-Samuelson theorem cannot be extended in a direct way to the joint-products case. Suppose now that our economy is a small, open one, facing given terms of trade $p > 1$ and consuming only commodity 2. Production will take place on CDE, in Fig. 1, and commodity 2 will be imported. Consider the effect of a non-prohibitive *ad valorem* tariff, at rate $t < (p-1)$, on imports of 2. That effect is, of course, equivalent to a reduction in the domestic value of p, and thus to a movement back *down* the wage-rent frontier in Fig. 2. The tariff has lowered *both* the real wage rate and the real rent rate. Whichever factor is thought to be used relatively intensively in the production of commodity 2, its real return is thus *lowered* by the imposition of the tariff on imports of commodity 2.

[1] Note that, since both the preference map and the consumption proportions differ between the two situations compared, there is no obvious 'welfare-based' standard of value.

(5) *Magnification Effects*

The Rybczynski and Stolper–Samuelson theorems present special cases of the 'magnification effects' which play such a large role in some areas of economic theorising,[1] most obviously those in which the (2×2) general equilibrium model provides the essential framework. It is thus implicit in the preceding two paragraphs that the familiar results concerning magnification effects cannot be extended in a direct way to models involving joint production.

VI. GENERALISATION

Consider an economy in which m primary inputs are used in n linear production processes to produce n commodities. Let the $(1 \times n)$ row vector $\mathbf{a}_i$ $(i = 1, \ldots, m)$ represent the quantities of primary input i used in the various processes at unit level: let the $(n \times n)$ non-singular matrix $\mathbf{N}$ represent, in its jth column, the *net* outputs of commodities from process j, operated at unit level. (Note that produced means of production are allowed for.) If $\mathbf{p}$ is the $(1 \times n)$ row vector of commodity prices and w_i the price of the ith primary input then, in the absence of interest payments,

$$\mathbf{pN} = \sum_1^m \mathbf{a}_i w_i$$

or

$$\mathbf{p} = \sum_1^m (\mathbf{a}_i \mathbf{N}^{-1}) w_i.$$

Hence if the semi-positive $(n \times 1)$ column vector of commodities, $\mathbf{z}$, is chosen as the standard of value, $\mathbf{pz} = 1$ and

$$\sum_1^m (\mathbf{a}_i \mathbf{N}^{-1} \mathbf{z})\, w_i = 1. \tag{1}$$

(1) is, of course, the 'factor price frontier' and it is clear that the necessary and sufficient condition for $(\partial w_i / \partial w_j) > 0$ is

$$(\mathbf{a}_i \mathbf{N}^{-1} \mathbf{z})\,(\mathbf{a}_j \mathbf{N}^{-1} \mathbf{z}) < 0. \tag{2}$$

Now in any viable single-products system, $\mathbf{N}^{-1}$ is semi-positive and thus (2) cannot hold; the presence of at least one joint-products process is a necessary (though not sufficient) condition for (2). Furthermore, the activity vector $\mathbf{x}$, required to produce a net output of $\mathbf{z}$, is given by $\mathbf{x} = \mathbf{N}^{-1}\mathbf{z}$, and if $\mathbf{x}$ is semi-positive (2) is again impossible. In other words, a necessary (though not sufficient) condition for (2) is that there be joint production *and* that w_i and w_j be measured in terms of a commodity bundle not producible (ignoring

[1] See Jones (1979, ch. 3). It will be understood that no criticism is implied of the theorems referred to or of their authors. (Thus Jones, for example, provides clear warnings that familiar results for single-product systems cannot be expected to apply directly to joint-product systems; see *ibid.*, ch. 8 *passim* and pp. 53, 314, 323.) It may be noted that valid joint-products theorems may often be obtained from valid single-products theorems by deleting 'commodity prices' and 'net outputs' from these latter and replacing those terms by 'process revenues' and 'process activity levels' respectively: the difficulties arise in framing joint-products theorems which *do* refer to 'commodity prices' and 'net outputs'.

disposal) with the processes under consideration. If, finally, $\mathbf{N}^{-1}\mathbf{z}$ *is* a vector with one or more negative elements, (2) will or will not hold depending on the values of $\mathbf{a}_i$ and $\mathbf{a}_j$. (Conversely, for given $\mathbf{a}_i$, $\mathbf{a}_j$ and non-semi-positive $\mathbf{N}^{-1}$, condition (2) makes it clear why the sign of $(\partial w_i/\partial w_j)$ can vary as the standard of value, $\mathbf{z}$, is varied.)

Suppose now that the market price of commodity i, p_i, is $(1+t_i)$ times the cost of production of i, due to the imposition of a sales tax at rate t_i. Then, if $\hat{\mathbf{t}}$ is the $(n \times n)$ diagonal matrix of such tax rates,

$$\mathbf{p}(\mathbf{I}+\hat{\mathbf{t}})^{-1}\mathbf{N} = \sum_{1}^{m}\mathbf{a}_i w_i$$

and the 'factor price frontier' becomes

$$\sum_{1}^{m}[\mathbf{a}_i\mathbf{N}^{-1}(\mathbf{I}+\hat{\mathbf{t}})\mathbf{z}]w_i = 1. \tag{1'}$$

If all the t_i are uniform, (1′) is just a radial contraction of the no-tax frontier (1). But if they are not, (1′) can be thought of as a no-tax frontier in which the standard of value $\mathbf{z}$ has been replaced by the standard $(\mathbf{I}+\hat{\mathbf{t}})\mathbf{z}$. Now, even if $\mathbf{z}$ is a producible commodity bundle, $(\mathbf{I}+\hat{\mathbf{t}})\,\mathbf{z}$ need not be, when $\mathbf{N}^{-1}$ is not semi-positive. It then follows that $(\partial w_i/\partial w_j) > 0$ is possible *on the with-tax frontier*, even when $\mathbf{z}$ is producible (and $(\partial w_i/\partial w_j)$ is thus necessarily negative on the no-tax frontier).

In the numerical example we deliberately ruled out produced means of production and interest payments in order to emphasise that the possibility of an upward-sloping wage–rent frontier stems from the presence of joint production alone, being independent of 'capital theory' phenomena. If we allow for a positive interest rate r, however, we need only modify the above remarks by noting that $\mathbf{N}$ is now to be interpreted as the gross output matrix *minus* $(1+r)$ times the input matrix, so that $\mathbf{N} \equiv \mathbf{N}(r)$, where every n_{ij} is either constant or decreasing with respect to r. Thus condition (2) becomes

$$[\mathbf{a}_i\mathbf{N}(r)^{-1}\mathbf{z}][\mathbf{a}_j\mathbf{N}(r)^{-1}\mathbf{z}] < 0 \tag{2'}$$

and the sign of $(\partial w_i/\partial w_j)$ may now depend on the level of the interest rate.

It may be remarked, finally, that if the *only* element of joint production involved in the system is that fixed capital is used and thus 'old' machines appear as joint products, then $(\partial w_i/\partial w_j) > 0$ can almost be ruled out. In such a system the only 'commodities' which cannot be produced alone are old machines. It then follows from what was said above that $(\partial w_i/\partial w_j) > 0$ could arise only if w_i, etc., were measured in terms of some old machine or, at the very least, in terms of a composite commodity in which old machines featured prominently. Hence, short of someone's presenting a plausible case for so measuring primary input prices, it may be said that *pure* joint production is necessary for $(\partial w_i/\partial w_j) > 0$; the presence of fixed capital does not suffice.[1]

[1] It must be noted carefully that if used machines are transferable between processes, or if any process uses more than one kind of machine, then the presence of fixed capital introduces *pure* joint production in the relevant sense: our statement in the text must therefore be interpreted in a strict way.

VII. CONCLUSION

A simple numerical example has sufficed to show that, even in the absence of produced means of production and of interest payments, the presence of even one joint-products process makes it impossible to assert *a priori* that the real wage–real rent frontier must be downward sloping. It has also been seen that, with joint production present, the very nature of the wage–rent frontier can depend on the standard in which wages and rents are measured; that some comparative statics results must be stated in a more qualified way than is necessary in single-product systems; that a tariff on an import can 'harm' *both* factors; and that an increase in the supply of a factor can, at constant commodity prices, lead to an increased output of *both* commodities. It was also indicated how these conclusions may be generalised.

In some branches of economic theory – for example, abstract general equilibrium theory, von Neumann growth theory – the presence of joint-products processes is always allowed for, yet in many others there is a strong tendency to analyse the single-products case exhaustively and then, at best, to mention briefly some of the effects of allowing for joint products. Such an approach appears to be unsatisfactory. First, because the presence of even one joint-products process, producing just two commodities, in an otherwise single-product system can alter qualitatively some of the most important properties of the system as a whole. Secondly, because pure joint-product processes are by no means rare; they are indeed very common and any casual impression to the contrary probably arises only from thinking of commodities and processes in a way which is far too aggregated to be relevant to the analysis of commodity prices and of the choice of production methods. *Joint production should be treated as the norm* and single-product systems and their special properties be pointed out as particular cases.

University of Manchester

Date of receipt of final typescript: November 1981

Reference

Jones, Ronald W. (1979). *International Trade: Essays in Theory.* Amsterdam: North Holland.

[10]

Joint Production and Technical Progress

Ian Steedman

It is widely believed and quite possibly true that technical progress is of great importance and it is certainly true, if less widely recognized by economic theorists, that joint production is very far from being exceptional[1]. The purpose of this paper is therefore to consider how the presence of joint production affects certain familiar results of the neo-classical theory of the consequences of technical change, in particular the clearcut results obtained in the context of a two commodity, two primary input model. It might perhaps seem obvious that technical *progress* must necessarily move the "factor price frontier" outwards from the origin, when the only "factor returns" involved are those to primary inputs, but it will be shown that this is not so in the presence of joint production, even when there are no produced means of production. This will be shown by means of a simple numerical example but the results will then be presented in more general form in a subsequent section.

1. THE CONVENTIONAL RESULTS

It may be helpful first to sketch the background to what follows, by reviewing the familiar neo-classical analysis of the effects of Hicks-neutral technical change in one sector, in the context of a two-sector, two-factor model[2]. The two commodities may be labelled 1 and 2 and the two factors

[1] Cf. I. STEEDMAN, "The Empirical Importance of Joint Production", *Manchester Discussion Paper in Economics*, Number 31, 1982; "L'importance empirique de la production jointe", in C. BIDARD (ed.), *La production jointe*, Paris, Ed. Economica, 1984.

[2] These results are set out in many standard works; for one good textbook example, see M. CHACHOLIADES, *International Trade Theory and Policy,* Tokyo, McGraw-Hill Kogakusha Ltd., 1978, pp. 349-358. The classic reference is, of course, R. FINDLAY and H. GRUBERT, "Factor Intensities, Technological Progress and the Terms of Trade", *Oxford Economic Papers*, XI, 1959, pp. 111-121.

called land and labour. It is assumed that, say, commodity 1 is unambiguously the land-intensive commodity, *i. e.*, that at every ratio of rents to wages, the land-labour ratio will be higher in sector 1 than in sector 2. It readily follows that, if p_i is the price of commodity i, W is the rent rate and w the wage rate, then (p_1/p_2) is a monotonically increasing function of (W/w).

Suppose now that sector 1 experiences Hicks-neutral technical progress, *i. e.* that the land-labour unit isoquant in sector 1 contracts towards the origin in a radial fashion. (Or, equivalently, that the "factor price frontier" relating $[W/p_1]$ to $[w/p_1]$ expands outwards from the origin in a radial fashion). It will be clear that (p_1/p_2) falls for every given (W/w) or, in other words, that (W/w) rises for every given (p_1/p_2). Since the whole (p_1/p_2) versus (W/w) curve is different before and after the progress, we have two alternative bases for comparing pre- and post-progress real rent rates and real wage rates; we may *either* hold (p_1/p_2) constant in making our comparisons *or* hold (W/w) constant. The usual neo-classical procedure is, in fact, not to choose between these two alternative bases but simply to make both sets of comparisons.

Consider first the constant (W/w) comparisons. Both (W/p_1) and (w/p_1) will have risen — and risen precisely by the rate of technical progress. (W/p_2) and (w/p_2), on the other hand, will be unchanged, since the "factor price frontier" relating them will be unchanged. Now consider the constant (p_1/p_2) comparisons. As has already been noted, if (p_1/p_2) is to be constant, (W/w) must be higher in the post-progress case; it follows at once that (W/p_2) will be higher and (w/p_2) will be lower after progress, since their frontier is unchanged. And it then follows in turn that (W/p_1) will be higher and (w/p_1) will be lower, since $(W/p_1) \equiv (W/p_2)\,(p_2/p_1)$, etc.

To summarise, the conventional results are as follows: Hicks- neutral progress in the production of the land-intensive commodity will, at constant (W/w), *raise* real rents and real wages (unless they are measured exclusively in terms of the other commodity) and will, at constant (p_1/p_2), *raise* real rents and *lower* real wages[3].

2. A TWO FACTOR, TWO COMMODITY EXAMPLE

We may now consider a very simple example of an economy using constant returns to scale processes, in which homogeneous land and homogeneous labour are used to produce two commodities. There are no produced means of production and rents and wages are paid ex-post, so

[3] Cf., e. g., R. N. BATRA, *Studies in the Pure Theory of International Trade*, London, Macmillan, 1973, p. 147.

that there are no interest payments. The common neo-classical assumptions of free disposal of commodities and of zero factor prices for factors less than fully employed will both be made. This model of production (without technical progress) has been set out before[4], but it will perhaps be helpful to the reader to present it in full once again.

TABLE 1

Process	*Labour*		*Land*		*Commodity 1*		*Commodity 2*
P_1	4	+	5	→	0	+	5
P_2	1	+	1	→	1	+	1
P_3	1	+	3	→	2	+	0
P_4	5	+	9	→	8	+	0

Table 1 shows the four available processes of production; P_1 produces only commodity 2, P_2 is a joint products process and P_3 and P_4 produce onnly commodity 1. If the fixed supplies of labour and land are 8 units and 12 units respectively, then the production possibility frontier is as shown in Figure 1. At A only P_1 is used and there is unused land. Along

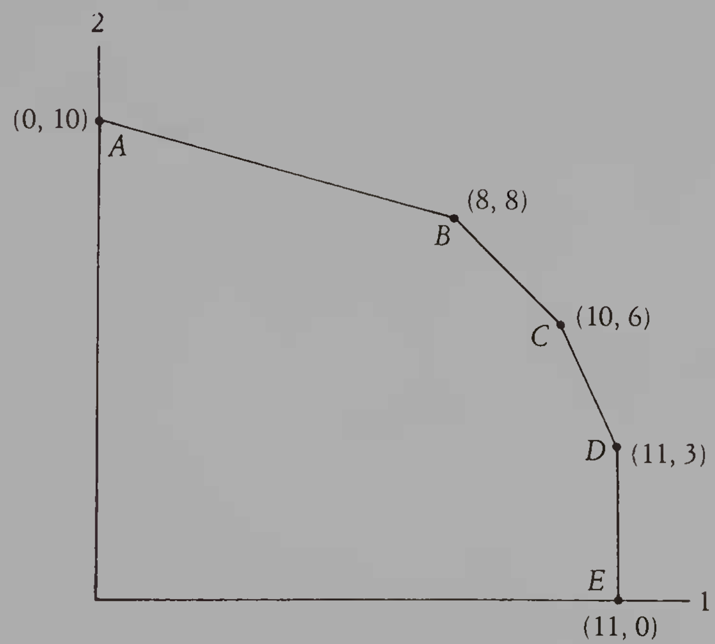


Fig. 1

[4] See I. STEEDMAN, "Joint Production and the Wage-Rent Frontier", *Economic Journal*, XCII, 1982, pp. 377-385.

AB both P_1 and P_2 are used but land is still not fully employed, since the ratio in which it is available (12/8) exceeds the ratio in which it is required in either process. At B itself, P_2 is used alone and land is again underemployed, for the same reason as before. Along BC and at C itself, however, P_2 and P_3 are used and *at* C (though not on BC) *both* land and labour are now fully utilised. This full utilisation of both factors obtains, in fact, all along CDE. On CD processes P_2, P_3 and P_4 are all in use, while at D itself, along DE and at E, processes P_2 and P_4 are used. (In every case, one draws the *separate* land and labour constraints for each combination of processes and then takes the "most binding" constraint, at any given ratio of the two outputs, as being part of the production possibility frontier $ABCDE$). The section DE is vertical because production is at D, while varying amounts of commodity 2 are freely disposed of. Since it is only on CDE that *both* land and labour are fully employed, and since the commodity price ratio is infinite along DE, we shall focus our attention on the section CD and on those commodity prices — $(p_1/p_2) > 1$ but finite — which lead competitive producers to be on CD.

In Figure 2 we show three alternative real rent-real wage frontiers; in each case W is the rent rate and w the wage rate, the sections "*c*" and "*d*" correspond to the corners C and D in Figure 1 and the arrows show the direction of movement as (p_1/p_2) rises. Figure 2 (a) is the rent-wage frontier when commodity 1 is the standard of value; Figure 2 (b) is the frontier when $p_1 + 6p_2 = 1$ defines the standard; and Figure 2 (c) is the frontier when commodity 2 is the standard. To see how these frontiers are obtained consider, for example, section "*c*" in Figure 2 (c). At corner C processes P_2 and P_3 are used, so we see from Table 1 that:

$$w + W = p_1 + p_2$$

and

$$w + 3W = 2p_1 \qquad [1]$$

It follows at once that:

$$W = -2p_2 + w$$

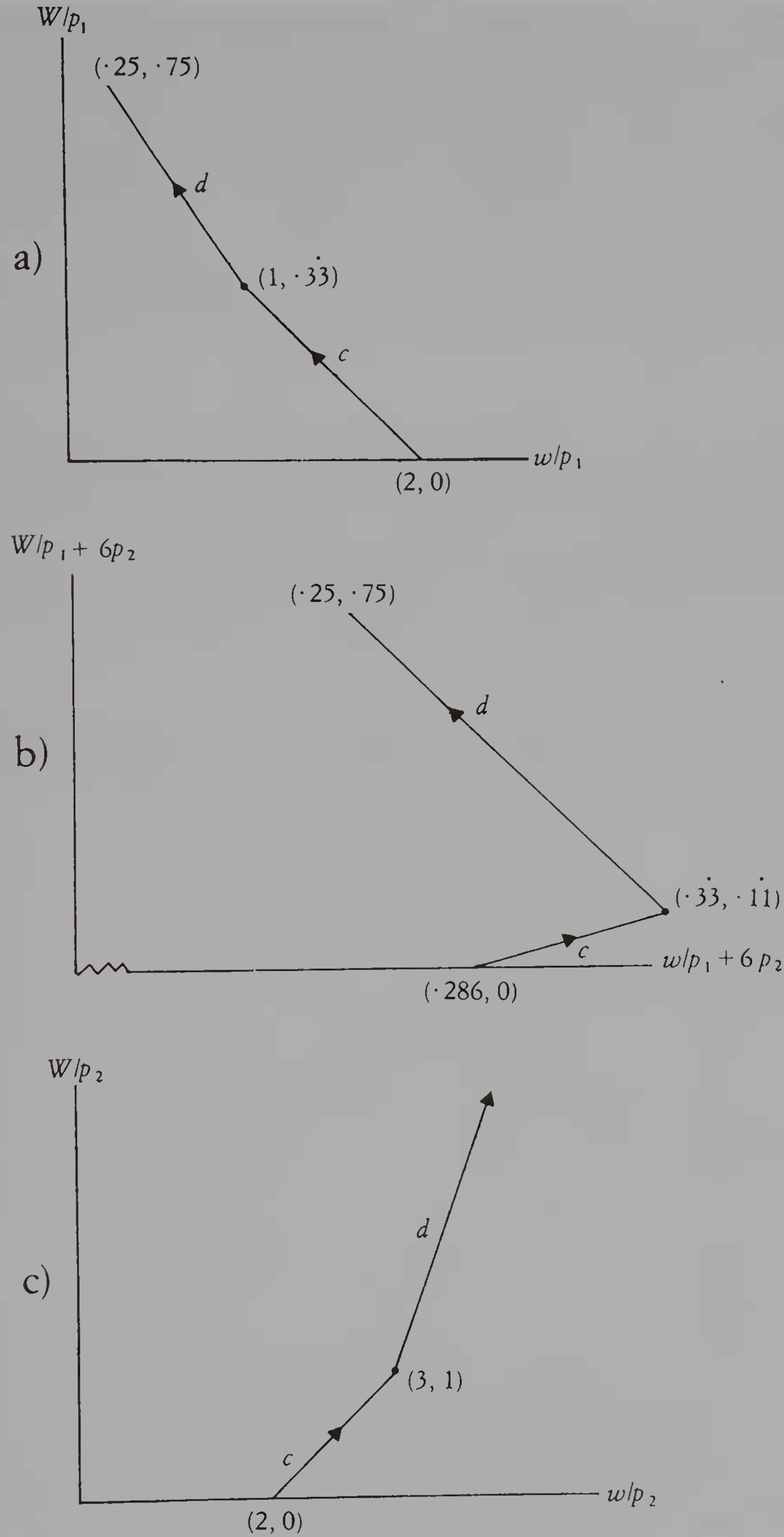
or

$$(W/p_2) = -2 + (w/p_2), \qquad [2]$$

which is the equation of "*c*" in Figure 2 (c). Moreover, from [1] and [2], $(p_1/p_2) = [(w + 3W) / (w - W)]$, so that (p_1/p_2) rises as (W/w) rises. All the other branches of the frontiers shown in Figures 2 are obtained in a similar way. (Many of the figures have been rounded off).

It is, of course, a striking feature of Figures 2 that not all sections of these real rent-real wage frontiers are downward sloping. Such sections

Fig. 2


W/p1
(·25, ·75)
d
a)
(1, ·33)
c
w/p1
(2, 0)
W/p1 + 6p2
(·25, ·75)
d
b)
(·33, ·11)
c
w/p1 + 6 p2
(·286, 0)
W/p2
d
c)
(3, 1)
c
w/p2
(2, 0)

45

contrast sharply with the necessarily downward sloping frontiers obtained from single-products systems and it has been shown how upward sloping frontiers can upset familiar comparative statics results concerning changes in demand, labour supply and real wages, and the Rybczynski and Stolper-Samuelson theorems[5]. It will be noted, however, that everywhere in Figures 2 (p_1/p_2) is positively related to (W/w). This constant feature suggests that commodity 1 may be thought of as the land-intensive commodity (and 2 as the labour-intensive one), in line with the standard result concerning relative factor intensities, and the corresponding movements of relative commodity prices and relative factor prices. (See above, first section).

Technical Progress. We now consider the effects of technical progress and shall concentrate on the case of "neutral" tecnical progress in processes P_3 and P_4, which constitute the unambiguous "commodity 1 sector". (Since P_2 also produces commodity 1, in addition to commodity 2, it could, of course, be suggested that we are not allowing for progress in the "full" sector 1. But P_2 cannot be classified unambiguously and, indeed, it is not self-evident how the concepts of neutrality, bias, etc. should be generalized to the case of joint production. See further below, however). Suppose that, with unchanged inputs, the outputs from P_3 and P_4 increase to $(2.2+0)$ and $(8.8+0)$, respectively, representing 10% neutral progress in the unambiguous sector 1. The new production possibility frontier is

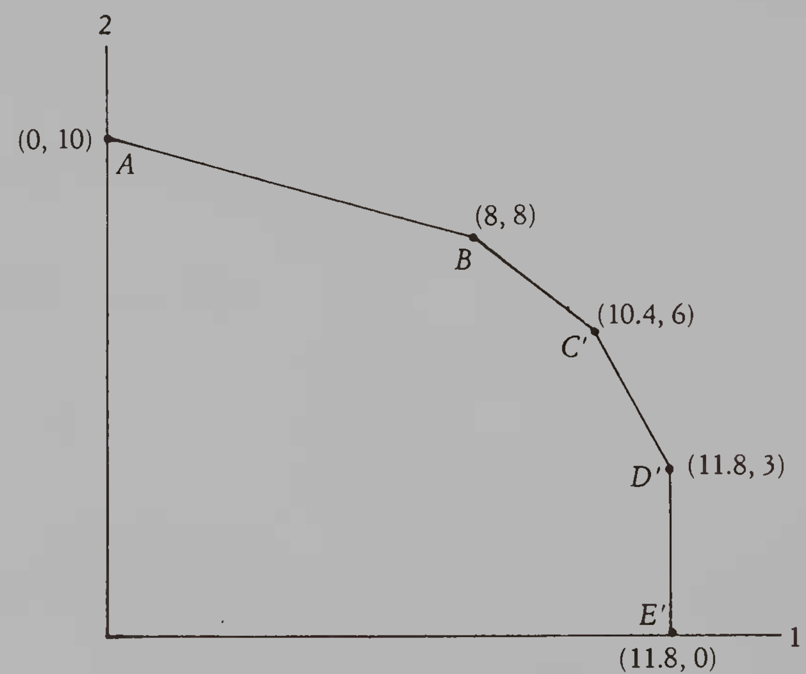


Fig. 3

[5] Cf. my "Joint Production", *op. cit*

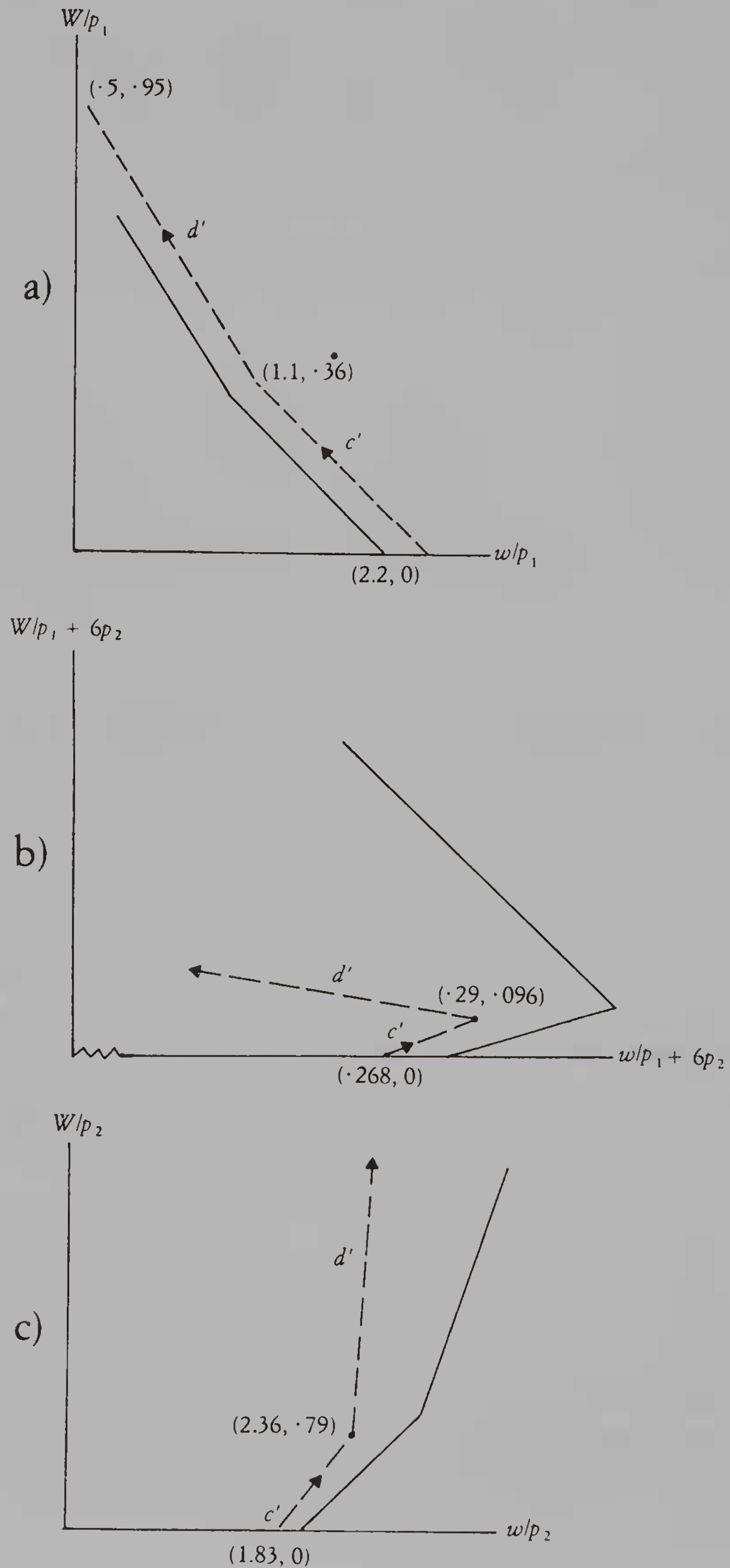
shown in Figure 3, where corners A and B are as in Figure 1 but corners C', D' and E' have all moved horizontally to the right, as compared with C, D and E in Figure 1. It does *not* follow, however, that every real rent-real wage frontier has moved out from the origin. Figure 4 reproduces Figure 2 in its solid lines and shows also, in dashed lines, the frontiers *after* the 10% progress in P_3 and P_4. In Figure 4 (a) the progress has indeed moved the frontier outwards but in Figures 4 (b) and 4 (c) technical *progress* has moved the frontiers towards the origin. (The new frontiers are, of course, derived in just the same way as the old ones).

In all three cases, the rent-wage *ratio* at the switch from "*c*" to "*d*" is unchanged by the progress. (This would indeed hold for any standard of value). But in Figures 4 (b) and 4 (c) the absolute values of *both* the rent and the wage have fallen at each rent-wage ratio. This result is quite contrary to the standard neo-classical theory of technical progress in the 2×2 model. (On the other hand, at any given (p_1/p_2) it can be shown that the real rent has risen and the real wage has fallen, which is entirely in line with that standard theory, if commodity 1 is taken to be the land-intensive commodity, as suggested above).

Further Cases. It has already been noted that, in the presence of joint product processes, it is not self-evident how one should generalise the usual neo-classical 2×2 analysis of technical change. In the above example only P_3 and P_4 — the unambiguously sector 1 processes — were subject to 10% technical progress. Suppose now, by contrast, that in addition to those changes, the output of commodity 1 from process P_2 also increases by 10%, the inputs and the output of commodity 2 being unchanged. In this new case, the real rent and real wage will both increase, as a result of the progress, at every rent-wage ratio (unless commodity 2 is the standard), as in standard theory. At constant (p_1/p_2), however, we do *not* obtain the conventional result, for it is readily seen that, on both "*c*" and "*d*", the real wage increases as a result of the technical progress (while the standard theory says that only the rent will increase, the wage falling). Thus whichever way we interpret "progress in sector 1", at least one of the conventional results fails to carry over to the joint products case.

(In order to consider "biased progress" it is, of course, appropriate to change the inputs and not the outputs of Table 1, reducing the land and labour inputs, in different proportions, in the relevant processes. It is clear from continuity considerations that "biased" input reductions will *not* necessarily restore all the conventional findings and it is left to the interested reader to construct suitable examples. Rather than present such examples, it is more interesting to note here that, in the presence of joint product processes, one has to consider "output neutrality or bias", as well as "input neutrality or bias", when defining types of technical change).

Fig. 4



W/p_1
(·5, ·95)
d'
a)
(1.1, ·36)
c'
w/p_1
(2.2, 0)
$W/p_1 + 6p_2$
b)
d'
(·29, ·096)
c'
$w/p_1 + 6p_2$
(·268, 0)
W/p_2
d'
c)
(2.36, ·79)
c'
w/p_2
(1.83, 0)

3. INTUITION

Having checked all the calculations and confirmed the results obtained from the above examples, the reader may still say, "Yes, these results are correct — but *why* can a real rent-real wage frontier slope upwards and *why* can technical progress move such a frontier inwards towards the origin?". The purpose of the present section is to help to reduce the implied sense of puzzlement.

Consider, for example, corner C in Figure 1 and branch "c" in Figure 2 (c), where processes P_2 and P_3 are in use, and try the experiment of "imputing" to commodities 1 and 2 the total amounts of labour and of land required to produce them. Let l_i (L_i) be the amount of labour (land) in question for commodity i. From Table 1 we see that:

$$l_1 + l_2 = 1$$

and

$$2l_1 = 1,$$

so that $l_1 = l_2 = (1/2)$. In the same way, we have:

$$L_1 + L_2 = 1$$

and

$$2L_1 = 3,$$

so that $L_1 = (3/2)$ and $L_2 = -(1/2)$. We have imputed to commodity 2 a *negative* amount of land required in its production. (So that $[L_1/l_1] > [L_2/l_2]$ — the condition that commodity 1 be the more land-intensive one — holds with a vengeance). But the reciprocal of L_2 is the value of (W/p_2) when $w = 0$ — and this is now seen to be negative! The upward sloping frontier, in terms of commodity 2, should now begin to seem less strange. Moreover, once it is realized that a joint production system may impute a *negative* amount of land use to some commodity, it will be seen that one ought to have no confident *a priori* expectations as to how technical progress will affect real wages and real rents. Our examples are really *not* surprising at all.

4. GENERALIZATION

In the above examples produced means of production were deliberately excluded, in order to emphasize that the possibility of non-neo-classical consequences of technical progress derives from joint production as such, and not from the interaction of joint production with capital theoretic complications. But we may now consider a system using n processes to produce n commodities by means of m types of primary input and inputs of the n types of commodity, there being a uniform rate of interest on the value of these latter. If the jth columns of B, A, E represent the outputs from, produced inputs to, and primary inputs to the jth process, at the unit level of operation, then:

$$wE = p\,[B - (1 + r)A];\; ps = 1 \qquad [3]$$

where w and p are row vectors of primary input "wage rates" and commodity prices, respectively, r is the interest rate, and s is a column vector representing the (composite commodity) standard of value. From [3]:

$$dwE = -\,wdE + dp\,[B - (1 + r)A] + p\,[dB - (1 + r)\,dA] \qquad [4]$$

and

$$dps = 0 \qquad [5]$$

if r is constant.

Case 1. If relative commodity prices are held constant, so that $dp = 0$, it follows from [4] that:

$$dwE = [-\,wdE - (1 + r)\,pdA + pdB] \qquad [6]$$

Any fall in E or A, and any rise in B, will increase the *RHS* of [6] but the effect on w depends, of course, on the structure of E. More specifically, let dE and dA both be zero, as in our numerical examples, so that:

$$dwE = pdB$$

If improvement is uniform *within any given process,* then $dB \equiv B\hat{t}$, for some diagonal matrix $\hat{t}$, and thus

$$dwE = (pB)\,\hat{t} \qquad [7]$$

It is clear from [7] that the presence of joint product processes — that is, the non-diagonal nature of B — can lead to no qualitative difference from the usual single product theory results. If improvement is uniform *for each given commodity,* however, $dB \equiv \hat{T}B$, for some diagonal matrix $\hat{T}$, and thus:

$$dwE = p\hat{T}B. \qquad [8]$$

It is clear from [8] that the non-diagonal nature of B *can* now lead to results which differ from the single product theory results. Thus both [7] and [8] confirm what was found above in our examples.

Case 2. If relative primary input prices are held constant, so that $dw = kw$ for some scalar k, it follows from [3], [4] and [5] that:

$$k = [-(wdEx_s) - (1 + r)(pdAx_s) + (pdBx_s)] \qquad [9]$$

where $x_s \equiv [B - (1 + r)A]^{-1} s$. In words, x_s is the (hypothetical) activity vector required to produce the standard bundle s for consumption and to maintain steady growth at a rate equal to r. If $x_s \geq 0$ then the *RHS* of [9] rises with every fall in E or A and with every rise in B. But if x_s contains one or more negative elements — *i. e.* the processes (B, A, E) cannot produce s and maintain growth at rate r — then k may respond "perversely" to some changes in (B, A, E)[6]. More specifically, suppose once more that dE and dA are zero, so that [9] becomes:

$$k = (pdBx_s)$$

In the *process* improvement case defined above:

$$k = (pB)\hat{t}x_s \qquad [10]$$

and joint production, since it can give rise to an x_s with negative elements, can involve k responding "perversely" to some elements of $\hat{t}$ in [10]. In the *commodity* improvement case, however, we have:

$$k = p\hat{T}Bx_s$$

or

$$k = p\hat{T}q_s \qquad [11]$$

where q_s is the *gross output* vector required to support s for consumption and growth at rate r. In our example A was zero and hence $q_s \equiv s$; in this case [11] shows that k must respond positively to the elements of $\hat{T}$, as stated above. More generally, however, the (hypothetical) vector q_s could have some negative elements, provided that the corresponding rows of Ax_s, the (hypothetical) capital stock vector, were also negative. Only if this is so can *commodity* improvement fail to raise all elements of w, when

[6] If x_s contains one or more negative elements it does *not* follow that s is an uninteresting standard in which to measure real primary input prices: see *ibid.*, pp. 380-381.

those elements are held in fixed proportions to one other. Again, then, [10] and [11] confirm our earlier findings.

It will be clear to the reader who has followed the arguments so far that we may develop [6] and [9] in various ways (for example, by writing $dB = \hat{T}_1 B\hat{t}_1$, $dA = \hat{T}_2 B\hat{t}_2$, $dE = -\hat{T}_3 E$) and that no very general results can then be expected. It can still be said, however, that the presence of joint production, by giving rise to a non-diagonal B and to the possibility of a non-semi-positive x_s, does mean that definite results are even harder to obtain than in the (very special) single products case.

5. CONCLUSION

It has been seen that, when joint production is allowed for, technical *progress* can actually move the primary-input-price frontier *inwards* towards the origin, when certain standards of value are used for measuring real wages, rents, etc. Moreover, familiar neo-classical results concerning the effects of technical progress, at either constant relative commodity prices or constant relative primary input prices, are no longer valid. Since joint production is so very widespread in real economies[7], these findings suggest that the standard neo-classical theory of the consequences of technical change is of little value. The challenge is, of course, to create an alternative and superior theory, able to take the fact of joint production in its stride.

Faculty of Economic and Social Studies, University of Manchester

[7] Cf. note 1 above.

Part II
Further Developments

[11]

Schweizerische Zeitschrift für Volkswirtschaft und Statistik
Revue suisse d'Economie politique et de Statistique

Herausgegeben von der Schweizerischen Gesellschaft für Statistik und Volkswirtschaft
Redaktion / Rédaction: Prof. Dr. H. G. Bieri, Prof. Dr. P. Stocker
Publiée par la Société suisse de Statistique et d'Economie politique

Nr. 4 106. Jahrgang 106e année Dezember 1970 Décembre 1970

On the Maximum Number of Switches Between Two Production Systems*

By Krishna Bharadwaj
Clare Hall, University of Cambridge

Introduction

In his "Production of Commodities by Means of Commodities"[1] *Piero Sraffa* introduces a classification of commodities into 'basics' and 'nonbasics': A commodity is called 'basic' to the system of production[2] if it enters, directly or indirectly, into every other commodity in the system; a commodity which does not do so is called 'nonbasic'. Opinions have diverged as to the role and significance of this distinction. In the literature, production systems containing only basics are more in vogue[3]. In an article on *Sraffa's* book, Professor *Newman*[4] proposes to 'abandon' nonbasics, by omitting them from the system, in the interest of analytical simplicity; particularly, to avoid the possibility of negative prices which a system containing nonbasics may give rise to (see Appendix below). However the existence of nonbasics is an objective property of a system and while nonbasics may be ignored in a first approximation they will have ultimately to be taken into account. Other writers, while considering nonbasics, have adopted, instead of this distinction between commodities,

* I am indebted to *Piero Sràffa* for his detailed criticisms on this paper. My thanks are also due to *P. Garegnani, L. Pasinetti,* and *Joan Robinson* for their very helpful comments

[1] *Sraffa P.*, Production of Commodities by Means of Commodities, Cambridge University Press, 1960.

[2] A system of production (alternatively, a production system) producing n commodities is a set of n production methods, one for each and each producing a single commodity.

[3] The 'Leontief system' which is frequently used is one such system.

[4] *Newman P.*, Production of Commodities by Means of Commodities, Schweizerische Zeitschrift für Volkswirtschaft und Statistik 1962, p. 58–75.

another classification based on an abstract property of the 'technology matrix' as being 'decomposable' or 'indecomposable'. A matrix is called 'decomposable' when by suitable interchange of columns and rows it can be reduced to the form $\begin{bmatrix} A_1 A_2 \\ 0 \; A_3 \end{bmatrix}$ where A_2 and A_3 are square matrices and 0 is a zero matrix. A matrix which cannot be so reduced is called 'indecomposable'. Thus a system containing at least one nonbasic has a decomposable technology matrix while a system with all basics has an indecomposable one. However, as will be seen, the basic-nonbasic distinction, referring to commodities in a given economic system, uses directly more information about it and does so in a way that helps perceive the economic content of the distinction.

This paper discusses a problem relating to 'reswitching of methods of production', an issue at the centre of a current controversy. A 'reswitch' in the methods of production is said to occur when, of two methods of production, one which has ceased to be the more profitable because of a change in the rate of profit becomes again more profitable than the other as the rate of profit moves further in the same direction. We shall take up here the more specific question of the maximum number of switches between two production systems and incidentally note how part of the difficulty in the reswitching controversy arises from not taking into account the particular role played by nonbasics.

We consider a situation where a number of commodities are being produced in annual cycles. Each commodity is produced by a separate industry, i.e. there is no joint production. A system of production with a specified net output is composed of the methods of production for the commodities that form the net output as well as others which enter, directly or indirectly, into their production. There is one method for each commodity in a system. We then suppose that there is an alternative method for one of the commodities and an alternative system is formed characterised by the use of the alternative method for that commodity. The introduction of the alternative method could entail the use of new commodities while possibly dropping some others. The switch point between the two systems corresponds to the rate of profit at which the two alternative methods produce the commodity at the same price (that is, at the switch point, the wage rate as also the prices of the commodities produced in both systems are equal). We deal in Section I with the case where each system consists only of commodities basic to it. In Section II we take up the case where the two systems also include nonbasics and where they differ in the method for a commodity which is nonbasic to both. We also examine there whether nonbasics entering the value unit (in terms of which wages and prices are expressed) but not entering either of the alternative
410 methods influence the number of switching possibilities, as is sometimes im-

plied (see p. 422 below). A question that arises here concerns the conditions guaranteeing the positivity of prices in a system including nonbasics. In this connection, we have reproduced, in the Appendix, letters[5] that were exchanged between *Sraffa* and *Newman*, following *Newman's* article, referred to above, which throw light on this point. In Section III we consider the advantages of the basic-nonbasic distinction for the discussion of switching. Section IV contains the conclusions.

I

Production Systems Consisting of only Basics

Consider a production system A involving m commodities, all basics. Suppose that for one of the commodities an alternative method of production is known which entails the use of some new commodities while possibly dropping out some others. Suppose the alternative production system formed by replacing the former method by the latter, call it system B, has n commodities all basics to it and that the two systems A and B have s commodities common to both; so that there are $(m-s)$ commodities used exclusively in system A and $(n-s)$ commodities used exclusively in system B. For convenience we renumber the commodities so that $1, 2 \ldots s$ are the s commodities common to the two systems, $s+1, s+2 \ldots m$ are the $(m-s)$ commodities exclusive to system A and $m+1, m+2 \ldots m+n-s$ are the $(n-s)$ commodities exclusive to system B. A switch point from one system to the other would be found at the rate of profit at which the wage and the price of each of the s common commodities are equal in the two systems.

Assuming wages are paid at the end of each annual cycle we write the price equations for the two systems:

System A

$$
\begin{array}{llllll}
(a_{11}p_{1a} & + a_{21}p_{2a} & + \dots + a_{s1}\,p_{sa})\,\lambda & & \dots + a_{01}\,W_a & = p_{1a} \\
(a_{12}p_{1a} & + a_{22}p_{2a} & + \dots + a_{s2}\,p_{sa})\,\lambda & & \dots + a_{02}\,W_a & = p_{2a} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
(a_{1s-1}p_{1a} & + a_{2s-1}p_{2a} & + \dots + a_{ss-1}\,p_{sa})\,\lambda & & \dots + a_{0s-1}\,W_a & = p_{s-1a} \\
(a_{1s}\,p_{1a} & + a_{2s}\,p_{2a} & + \dots + a_{ss}\,p_{sa} & + a_{s+1s}\,p_{s+1s} & \dots + a_{ms}\,p_{ma})\,\lambda + a_{0s}\,W_a & = p_{sa} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(a_{1m}p_{1a} & + a_{2m}p_{2a} & + \dots + a_{sm}\,p_{sa} & + a_{s+1m}\,p_{s+1a} & \dots + a_{mm}p_{ma})\,\lambda + a_{0m}\,W_a & = p_{ma}
\end{array}
\tag{I}
$$

[5] I am grateful to Professor *Newman* and Mr. *Sraffa* for allowing me to publish these letters.

where $p_{1a}, p_{2a} \ldots p_{ma}$ are the prices of commodities $1, 2 \ldots m$ and W_a the wage rate in system A and $\lambda = 1 + r$ where r is the rate of profit; a_{ij} $(i, j = 1, 2, \ldots m)$ and a_{0j} $(j = 1, 2 \ldots m)$ are the commodity input and labour coefficients respectively for System A. An analogous notation to represent prices and the wage rate in the system B is adopted to write the price equations for the system B. (It would be noted that $i, j = 1, 2 \ldots s, m+1, \ldots m+n-s$ and $a_{ij} = b_{ij}$ for $j = 1, 2 \ldots s-1$.)

System B:

$$
\begin{array}{l}
(a_{11}\, p_{1b} + a_{21}\, p_{2b} + \ldots + a_{s1}\, p_{sb})\, \lambda + a_{01}\, W_b = p_{1b} \\
(a_{12}\, p_{1b} + a_{22}\, p_{2b} + \ldots + a_{s2}\, p_{sb})\, \lambda + a_{02}\, W_b = p_{2b} \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
(a_{1s-1}\, p_{1b} + a_{2s-1}\, p_{2b} + \ldots + a_{ss-1}\, p_{sb})\, \lambda + a_{0s-1}\, W_b = p_{s-1b} \\
(b_{1s}\, p_{1b} + b_{2s}\, p_{2b} + \ldots + b_{ss}\, p_{sb} + b_{m+1s}\, p_{m+1b} + \ldots \\
\qquad\qquad + b_{m+n-ss}\, p_{m+n-sb})\, \lambda + b_{0s}\, W_b = p_{sb} \\
(b_{1m+1}\, p_{1b} + b_{2m+1}\, p_{2b} + \ldots + b_{sm+1}\, p_{sb} + b_{m+1m+1}\, p_{m+1b} + \ldots \\
\qquad\qquad + b_{m+n-sm+1}\, p_{m+n-sb})\, \lambda + b_{0m+1}\, W_b = p_{m+1b} \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
(b_{1m+n-s}\, p_{1b} + b_{2m+n-s}\, p_{2b} + \ldots + b_{sm+n-s}\, p_{5b} + b_{m+1\, m+n-s}\, p_{m+1b} + \ldots \\
\qquad\qquad + b_{m+n-sm+n-s}\, p_{m+n-sb})\, \lambda + b_{0m+n-s}\, W_b = p_{m+n-sb}
\end{array}
\qquad \text{(II)}
$$

We need now to find out such values of λ at which $p_{ia} = p_{ib}$ $(i = 1, 2 \ldots s)$ when $W_a = W_b = W$[6]. We take $p_{1a} = p_{1b} = 1$ so that commodity 1 is chosen as numeraire. We have the problem of unequal numbers and different kinds of basics in the two systems (namely commodities $1, 2 \ldots m$ in system A and $1, \ldots s, s+1, \ldots m+n-s$ in system B) which affect the wage profit relations in the respective systems. We now introduce in system A, as nonbasics, the

[6] The condition regarding the equality of relative prices of the s commodities common to the two systems is equivalent to stating that the wage in terms of any of them should be equal, at the switch point, in the two systems. A point of some interest to note is that, if we consider any two production systems differing in the method of production for more than one commodity common to them and express the wage and prices in the two systems in terms of someone of the commodities common to them, the relative prices for these common commodities in the two systems may not necessarily be equal at all the intersections of the wage profit curves for the two systems. The equality of the relative prices would have to be laid down as *a priori* condition to obtain the switch
412 points among those points of intersections.

commodities which are exclusive to system B and thereby augment the matrix[7]; let the matrix so augmented be called system A^+. Similarly we construct B^+ from system B. These would appear as follows:

$$[A^+, A_0^+] =$$

$$\left[\begin{array}{ccc|cc|cc|c}
a_{11} & a_{21} & \cdots a_{s1} & & & & & a_{01} \\
a_{12} & a_{22} & \cdots a_{s2} & & & & & a_{02} \\
\vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & \vdots \\
a_{1s-1} & a_{2s-1} & \cdots a_{ss-1} & & & & & a_{0s-1} \\
\hline
a_{1s} & a_{2s} & \cdots a_{ss} & a_{s+1s} & \cdots a_{ms} & \multicolumn{2}{c|}{O} & a_{0s} \\
\hline
a_{1s+1} & a_{2s+1} & \cdots a_{ss+1} & a_{s+1s+1} & \cdots a_{ms+1} & & & a_{0s+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \vdots \\
a_{1m} & a_{2m} & \cdots a_{sm} & a_{s+1m} & \cdots a_{mm} & & & a_{0m} \\
\hline
b_{1m+1} & b_{2m+1} & \cdots b_{sm+1} & & & b_{m+1\,m+1} & b_{m+n-s\,m+1} & b_{0m+1} \\
\vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \vdots & \vdots & \vdots \\
b_{1m+n-s} & b_{2m+n-s} & \cdots b_{sm+n-s} & & & b_{m+1\,m+n-s} & b_{m+n-s\,m+n-s} & b_{0m+n-s}
\end{array}\right]$$

$$[B^+, B_0^+] =$$

$$\left[\begin{array}{ccc|cc|cc|c}
a_{11} & a_{21} & \cdots a_{s1} & & & & & a_{01} \\
a_{12} & a_{22} & \cdots a_{s2} & & & & & a_{02} \\
\vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & \vdots \\
a_{1s-1} & a_{2s-1} & \cdots a_{ss-1} & & & & & a_{0s-1} \\
\hline
b_{1s} & b_{2s} & \cdots b_{ss} & \multicolumn{2}{c|}{O} & b_{m+1s} & \cdots b_{m+n-s\,s} & b_{0s} \\
\hline
a_{1s+1} & a_{2s+1} & \cdots a_{ss+1} & a_{s+1s+1} & \cdots a_{ms+1} & & & a_{0s+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \vdots \\
a_{1m} & a_{2m} & \cdots a_{sm} & a_{s+1m} & \cdots a_{mm} & & & a_{0m} \\
\hline
b_{1m+1} & b_{2m+1} & \cdots b_{sm+1} & & & b_{m+1\,m+1} & b_{m+n-s\,m+1} & b_{0m+1} \\
\vdots & \vdots & \vdots & \multicolumn{2}{c|}{O} & \vdots & \vdots & \vdots \\
b_{1m+n-s} & b_{2m+n-s} & \cdots b_{sm+n-s} & \multicolumn{2}{c|}{\cdots} & b_{m+1\,m+n-s} & b_{m+n-s\,m+n-s} & b_{0m+n-s}
\end{array}\right]$$

[7] We do so since, at a switch point, all the methods in both systems must be competitive.

where A^+ and B^+ are augmented matrices of commodity input coefficients and A_0^+ and B_0^+ are augmented matrices of labour coefficients. The switch point values of λ which satisfy the condition $p_{ia}^+ = p_{ib}^+ = p_i^+$ and $W_a^+ = W_b^+ = W^+$ are to be obtained by solving a vector of polynomial equations given by

$$[F_i^+(\lambda)] = [A^+ - B^+]\,\lambda\,[p^+] + [A_0^+ - B_0^+]\,W^+ = 0 \qquad (1)$$
$$i = 1, 2 \ldots m+n-s.$$

where matrices A^+ and B^+ are both $(m+n-s) \times (m+n-s)$ and $p^+ = (1, p_2^+, p_3^+ \ldots p_{m+n-s}^+)$

There would be as many non-zero elements in this vector as the number of differing methods of production in the augmented systems and in this case, therefore, there is only one polynomial to be solved, namely, $F_{(s)}(\lambda) = 0$. In other words, it is sufficient to equate the price of the commodity s, the only commodity to have a different method of production in the two systems. We could solve for p^+ and W^+ in terms of λ in either of the two systems A^+ or B^+[8].

From the system A^+, we can write:

$$p^+ = [p_{ia}^+] = \frac{[J_i^+(\lambda)]}{[g^+(\lambda)]} \;(2) \text{ and } W_a^+ = W_b^+ = \frac{f^+(\lambda)}{g^+(\lambda)} \qquad (3)$$

where $J_{(i)}^+(\lambda)$, $f^+(\lambda)$ and $g^+(\lambda)$ are of at most $(m+n-s-1)$, $(m+n-s)$ and $(m+n-s-1)$ degree in λ respectively. Hence the polynomial function in (1) can have at most $(m+n-s)$ roots and hence the maximum number of switches between the two systems is $(m+n-s)$. As a polar case, if the two systems A and B have only one basic commodity common between them, the maximum number of switch points would be $(m+n-1)$[9].

Of the number of possible switches thus obtained as an upper bound, the economically relevant number of switch points would be obtained only after excluding repeated counting of repeated roots, complex roots and those lying beyond the range $1 \leqslant \lambda \leqslant 1+R$ where R is the lower of the two maximum rates of profit for the two systems[10]. Also, we consider only those situations in which all prices are positive[11].

[8] Since $p_{ia}^+ = p_{ib}^+ = p_i^+$ and $W_a^+ = W_b^+$ either of the two systems can be so used.

[9] A particular illustration of this is the result obtained by *Joan Robinson* and *K. A. Naqvi* (Quarterly Journal of Economics., p. 590) where they consider two production systems, one with wheat and iron as basics and another with wheat and aluminium as basics and obtain three switch points between them.

[10] These switch points could also include such cases where, at the switch point rate of profit, the wage-profit curve for one system is tangential to that of the other wholly from above: that is, the same system continues to be the more profitable one on both sides of the switch point.

414 [11] See p. 418–9 below.

The case of the same basics in the two alternative systems which is often treated as 'general'[12] is seen to be only a particular case of the above more general formulation. With all n basics common to the two systems the maximum number of switches is seen to be only n. The assumption that the two systems have the same numbers and kinds of basics while they differ in the methods of production is extremeley restrictive since it is unlikely that two different methods will use identical materials and tools.

The discussion of switching possibilities between two production systems has been usually conducted under the assumption that the two systems under consideration may differ in the methods of production for more than one (and up to all) commodities common to them. This has been described as the most general model; but, far from being a general case this would be a very exceptional one[13]. Switches would occur between systems differing in the method of production for only one basic commodity common to the two systems. When more than one basic common to them is produced by a different method in the two systems, it is clear from the condition for obtaining the switch point set out in (1) above that a set of polynomial functions in λ of that number would have to have at least one common root. Such a condition would be fulfilled only as a fluke[14].

[12] See, for example, *Bruno M.*, *Burmeister E.* and *Sheshinski E.*, The nature and implications of reswitching of techniques, Quarterly Journal of Economics 1966, p. 526–553.

[13] Analytically there is no loss of generality involved in a procedure of successive consideration of production systems using a different method of production for only one of the commodities common to them as, given all possible systems of production, it could not lead to any different outermost boundary of wage-profit curves. Incidentally, it would be noted that whatever be the number of commodities produced by different methods in the two systems the *maximum* number of switching possibilities would still be equal to the total number of distinct (without double counting) basics in the two systems together.

[14] Alternatively, we could arrive at the same conclusion by observing that *a* system with m basic commodities (such as A above) has $(m+1)$ unknowns (m–1 relative prices, wage and the rate of profit) and m independent equations to solve them. Hence one more additional equation can be accommodated to make the system determinate even though it does not bring in any additional commodity with its price. This additional equation would be the alternative method for one commodity in the system. If the alternative method brings in additional commodities there would have to be as many additional price equations. In our example above there are $(m+n-s)$ distinct commodities in the two systems together and $(m+n-s+1)$ independent methods would be needed to determine the prices, wage rate and the rate of profit. If more than one commodity in system A has a different method in system B the system of equations would be overdetermined. 415

II

Systems Including Commodities which are Nonbasics to Both

The production systems discussed so far involved commodities which were basic to one or the other system. We now turn to those which include commodities that are nonbasics to both. Even in such systems, when the two systems are characterised by a different method of production only for a *basic* produced in both, the switch points between the two methods for the basic (and hence the two systems) would be determined by solving for prices within the augmented systems; the latter would include *only* the methods of production for the commodities which are basics to one or the other system. The methods of production for commodities which are nonbasic to both systems can be ignored. We take up two cases where the methods of production of such nonbasics may not be so ignored.

1. If there are alternative methods of production for a nonbasic there would be switches in the method of production for that nonbasic as the rate of profit changes[15]. (Each of the basics has only one known method.)

2. Nonbasics may enter the value unit in terms of which prices and wage are expressed. It is evident that the nonbasics could enter the price equations of the basics only indirectly in this way. If prices are expressed as functions of the rate of profit, the maximum degree of the price equation for a basic would be given by the number of commodities entering directly or indirectly into this value unit which includes nonbasics. This seems to have suggested that the maximum number of possible switches is also given by that number (see p.418 below). We examine the question whether the nonbasics which enter the value unit, directly or indirectly, but do not enter, directly or indirectly, into either of the alternative methods between which switches are being considered, influence the switching possibilities. We first consider the question of the alternative methods for a nonbasic.

Alternative Methods for a Nonbasic: Suppose that one of the nonbasics in a system A has an alternative method of production. When the alternative method is used, it might entail the use of some nonbasics peculiar to itself while possibly dropping some others. Let us call the system characterised by the use of the latter method for the nonbasic system B. Suppose also that commodity 1 (basic to both systems) is numeraire. We can follow the same procedure as on page 413 above and write the augmented systems A^+ and B^+ which would now include, in addition to the methods of production for com-

[15] However the switches in the method of a nonbasic have to be clearly distinguished from those for a basic inasmuch as the former would not affect the relative prices of
416 the basics in the system or the maximum rate of profit whereas the latter do.

modities basic to at least one system also those for nonbasics to both which enter, directly or indirectly, into one or the other of the alternative methods of production for the nonbasic in question. Such commodities as are nonbasics to both systems and do not enter either of these alternative methods would not appear in the augmented systems. The augmented matrices would differ in only one row, namely that representing the method of production for the nonbasic with the alternative methods. The maximum number of switches for the two systems is given, as in the earlier case, by the dimension of the augmented matrix, i.e. by the number of distinct commodities, basic and nonbasic, without double counting, that enter in at least one of the alternative methods of production for the nonbasic[16].

Nonbasic Entering Value Unit. Suppose that there are alternative methods of production for only a basic and nonbasics enter the value unit. Further that these are nonbasics to both the systems, the two systems differing in the method for the basic[17]. Would the nonbasics entering the value unit affect the maximum number of switches between the two systems?

As a simple illustration we take production systems A and B each with two basics (designated commodities 1 and 2 in system A and 1 and 4 in system B) and one nonbasic (commodity 3 in both). They differ in the method for commodity 1. The nonbasic is produced by the same method in the two systems and forms the value unit. Representing the two systems as below:

System A

$$(a_{11}\, p_{1a} + a_{21}\, p_{2a})\, \lambda + a_{01}\, W_a = p_{1a}$$

$$(a_{12}\, p_{1a} + a_{22}\, p_{2a})\, \lambda + a_{02}\, W_a = p_{2a}$$

$$(a_{13}\, p_{1a} + a_{33}\, p_{3a})\, \lambda + a_{03}\, W_a = p_{3a}$$

[16] One may consider, as a curiosum, the case of a commodity basic to system A which when produced by an alternative method becomes itself a nonbasic in system B, each of the other commodities having only one known method. This would, however, imply that the two systems would have no commodities in common which are basic to both.

[17] This needs some clarification: Two production systems differing in the method of production for one of the basics common to them could have one or more commodities which are basic to one and not to the other. If wages and prices are expressed in terms of the 'standard commodity' of either one of the systems (for the definition of the 'standard commodity' see *Sraffa P.*, op. cit., p. 18–20) as *Sraffa* does (see op. cit., p. 85) or in terms of any value unit involving commodities exclusively basic to one of the systems, the other system will have its prices and wage expressed in terms of a value unit involving nonbasics to itself. These nonbasics are however basics to the other system and hence enter, directly or indirectly, into the production of the basic (with alternative methods) in that system.

System B

$$(b_{11}\, p_{1b} + b_{41}\, p_{4b})\, \lambda + b_{01}\, W_b = p_{1b}$$

$$(b_{14}\, p_{1b} + b_{44}\, p_{4b})\, \lambda + b_{04}\, W_b = p_{4b}$$

$$(a_{13}\, p_{1b} + a_{33}\, p_{3b})\, \lambda + a_{03}\, W_b = p_{3b}$$

With $P_{3a} = P_{3b} = 1$, the switch points are to be obtained by augmenting *A and* *B* to A^+ and B^+ respectively as discussed earlier and by solving the following polynomial:

$$(a_{11}-b_{11})\, \lambda\, p_{1a}^+ + a_{21}\, \lambda\, p_{2a}^+ + (-b_{41})\, \lambda\, p_{4a}^+ + (a_{01}-b_{01})\, W_a^+ = 0 \qquad (4)$$

where P_{1a}^+, P_{2a}^+, P_{4a}^+ and W_a^+ are themselves polynomials in λ.

In this simple case we can make the following observations:

1. If $a_{33} = 0$, the nonbasic does not use itself in its own production, the polynomial in λ in (4) above has the maximum degree only three and hence the maximum number of switch points is only three.

2. If $a_{33} \neq 0$ then $\lambda = 1/a_{33}$ happens to be the additional solution for λ. It would be noted, however, that at this value of λ, with the nonbasic as the value unit, the prices of the basic commodities can no more satisfy the positivity condition[18].

The observations hold even if a composite commodity consisting of basics and nonbasics (e.g. $q_1\, p_{1a} + q_2\, p_{2a} + q_3\, p_{3a} = 1$ with q_1, q_2, q_3 constants) is adopted as a value unit. If there are nonbasics which are required for the production of the nonbasic that enters the value unit we can generalise the above observations. It would be found that:

i) Such of the nonbasics that enter directly or indirectly into their own production and enter, directly or indirectly, into the value unit would add to the number of possible solutions to the polynomial equation given by (4) above[19].

[18] The price equation for the nonbasic at $\lambda = 1/a_{33}$ gives in system *A:*

$$\frac{a_{13}}{a_{33}}\, p_{1a} + \frac{a_{23}}{a_{33}}\, p_{2a} + a_{03}\, w = 0.$$

With $a_{33} > 0, a_{13}, a_{23} \geqslant 0$ this cannot be satisfied for positive prices. See also below p. 419.

[19] Thus consider two nonbasics in the above system (commodities 3 and 5) with the commodity 3 as a value unit and commodity 5 entering its production. We have the two systems differing in the method for commodity 1 as before. The price relations are given by (in system *A*):

$$(1-a_{11}\, \lambda)\, p_{1a} - a_{21}\, \lambda\, p_{2a} - a_{01}\, W_a = 0$$

$$-a_{12}\, \lambda\, p_{1a} + (1-a_{22}\, \lambda)\, p_{2a} - a_{02}\, W_a = 0$$

$$-a_{13}\, \lambda\, p_{1a} \quad - a_{53}\, \lambda\, p_{5a} - a_{03}\, W_a = a_{33}\, \lambda - 1$$

$$-a_{14}\, \lambda\, p_{1a} \quad + (1-a_{55}\, \lambda)\, p_{5a} - a_{04}\, W_a = a_{35}\, \lambda$$

ii) However the solutions that are added on are the rate(s) of profit equal to the rate of reproduction for the separate nonbasic or a group of interconnected nonbasics, as the case may be. These switch points however would have to be ruled out for the following reasons: If the rate of reproduction of a nonbasic (or a group of interconnected nonbasics) is smaller than the lower of the two maximum rates of profit for the two systems, the condition regarding the positivity of prices at those switch points will not be satisfied. In fact with the nonbasic as a value unit and a rate of profit equal to its rate of reproduction, one obtains, as *Sraffa* shows[20], a 'formal' solution in which "the price of every commodity is zero". Thus the two production systems could 'formally' have a switch point which has to be ruled out since we consider only those switch points at which all prices are positive.

More importantly, such a value of the rate of profit at the switch point might well fall beyond the maximum rate of profit for at least one of the production systems. A fuller discussion on this issue appears in the *Sraffa-Newman* correspondence which is reproduced in the Appendix below. *Sraffa* argues there that instances of a nonbasic in the system having a rate of reproduction less than the maximum rate of profit for the system would be hardly met with and that the particular example of beans, a nonbasic of that type, which he employed in Appendix B of his book had to be invented in order to establish that, with such a nonbasic in the system, positivity of prices could not hold at a rate of profit equal to the rate of reproduction of that nonbasic.

Propositions similar to i) and ii) above can be proved in the case where the commodity with the alternative methods is a nonbasic, each of the other commodities common to the two systems having the same method. If there are other nonbasics in the system, which, while not entering either of the alternative methods of production, directly or indirectly, enter the value unit,

For system B coefficients in the first and second equations alone are different, the respective price equations being:

$$(1-b_{11}\,\lambda)\,p_{1b} - b_{41}\,\lambda\,p_{4b} - b_{01}\,W_b = 0$$
$$-\,b_{14}\,\lambda\,p_{1b} + (1-b_{44}\,p_{4b}) - b_{04}\,W_b = 0$$

The switch points for the two systems are obtained as before by solving for λ as in (4) above. The expression on the left hand side of (4) gives in this case a common factor, a polynomial in λ of degree at most two and at most two additional values for λ (i.e. two more than would have been obtained if only basics formed the value unit). This common factor is $\{1-a_{33}\,\lambda)\,(1-a_{55}\,\lambda)-a_{35}\,\lambda\,a_{53}\,\lambda\}$ which when equated to zero gives the value of $r = \lambda - 1$, equal to the rate of reproduction of the group of nonbasics. It will be noted that if $a_{33} = 0$ and $a_{35} = (0$ with $a_{53} > 0$, given) then $\lambda = 1/a_{55}$. This is the case when only commodity 5 of the two nonbasics requires itself in its own production. Similarly if $a_{33} > 0$ but $a_{35} = 0$ and $a_{55} = 0$ then $\lambda = 1/a_{33}$. In both cases the solution for λ gives a rate of profit equal to the rate of reproduction of the separate nonbasics.

20 *Sraffa P.*, op. cit., Appendix B, p. 90–91.

directly or indirectly, then such nonbasics would not add to the maximum number of solutions for switch points between the two systems excepting in a formal way, as pointed out in ii) above.

III

In the foregoing we have used the classification of commodities into basics and nonbasics to discuss the question of switching possibilities between two systems. Another classification which has been used more frequently in current discussions is that of decomposable and indecomposable systems. However, given a production system, the classification of the commodities involved into basics and nonbasics uses more of the available information about the system than does the classification of that system as decomposable or indecomposable. By stating that the system contains (or does not contain) nonbasics we would have already implied that the system is decomposable (or indecomposable). The classification of commodities into basics and nonbasics would further inform us as to which commodities in that system give rise to its decomposability. For, by their very nature, the basics in the system can be identified as forming a wholly interconnected group (we shall call this system, formed by all the basics in the system, the Basic system) while the nonbasics cannot do so since, while they require basics for their production, they are not themselves required in the production of the basics.

The additional information incorporated in the basic-nonbasic distinction is relevant to the discussion of switching possibilities between systems since it directly leads onto a distinction between two types of switches which have different consequences. A switch in the method for a basic implies that the two systems (each characterised by the method that it uses for that basic) would have different Basic systems, each with a maximum rate of profit, different from that of the other. On the other hand, a switch in the method for a nonbasic does not affect the maximum rate of profit of the system nor the prices of the basics, i.e. the Basic system is not affected in any way by a change in the method for a nonbasic. Another instance of the asymmetry between the two classes of commodities can be seen in this, that propositions concerning the switches in a basic (such as the maximum number of possible switches and the rate of profit at which a switch occurs) and the transition from one system to another that these switches imply can be derived from the consideration of the Basic system alone, ignoring the nonbasics, while such propositions concerning a nonbasic cannot be based on the consideration of nonbasics
420 alone.

It would seem that some of the confusion which arose from a paper by *D. Levhari*[21] and which concerned the decomposability of a system and its relevance to the reswitching of techniques could have been avoided if the distinction between basics and nonbasics had been taken into account. *Levhari* in his paper claimed to have demonstrated that if there are n commodities in a system and if the i^{th} commodity ($i = 1, 2, \ldots n$) had k_i alternative methods of production (so that there are $\prod_{i=1}^{n} k_i$ possible systems of production) it is impossible that anyone system of production should switch back as the rate of profit continues to move in anyone direction. This claim was withdrawn later by *Levhari* and *Samuelson*[22]. They there explained that *Levhari's* original paper had accepted the possibility of a decomposable system's reswitching as 'established without question' by *Ruth Cohen, Joan Robinson* and *P. Sraffa* and it had attempted to show that such a reswitching could not happen in an indecomposable system. It should however be noted that *Sraffa's* demonstration of the possibility of reswitching was not limited to a decomposable case as *Levhari* had believed. When *Sraffa* takes the case of the alternative methods for a basic (having first considered briefly that of the alternative methods for a nonbasic) his argument does not require the existence of nonbasics in the system: the proposition concerning the possibility of reswitching of the basic holds whether the original system is decomposable or not. It is true that *Sraffa* makes a distinction between basic *uses* and nonbasic *uses* but this distinction is introduced only in order to facilitate comparison between the alternative methods within the *same* system at rates of profit at which the two methods are not equally profitable, i.e. away from the switch points. Each of the two commodities considered there (copper I and copper II) is basic to one or the other system. Off the switch point any comparison of the two alternative methods of producing copper at the prices of the system characterised by the use of copper I as basic implies treating the method producing copper II as a nonbasic in that system; the production matrix including *both* is decomposable[23].

[21] *Levhari D.*, A nonsubstitution theorem and switching of techniques, Quarterly Journal of Economics 1965, p. 98–105.

[22] *Levhari D.* and *Samuelson P.*, The nonreswitching theorem is false, Quarterly Journal of Economics 1966, p. 518–519. The *Levhari* Theorem was withdrawn when it was refuted conclusively by a number of writers (see *Pasinetti L., Morishima M., Garegnani P., Bruno M., Burmeister E.* and *Sheshinski E.*) in the Symposium on paradoxes in capital theory in the Quarterly Journal of Economics 1966, p. 504–583.

[23] *Levhari's* argument is based on the assumption that there are a number of alternative methods for producing each commodity and that each one of the possible systems of production consists of only basics. His statement on the nonreswitching of anyone system of production seems to have been suggested by the conjecture that when there is a wide range of known methods for each one of the several commodities the

The case where two production systems have different basics, such as considered in Section I above, is a parallel instance where decomposability in the process of a comparison between the two systems arises. Each one of the two systems A and B appearing there is indecomposable and yet a comparison of the two systems implies the use of augmented systems A^+ and B^+ formed by adding to each system as nonbasics, the commodities which are basics only in the other. As we have seen such nonbasics peculiar to one or the other augmented systems of Section I have to be distinguished from the nonbasics common to *both* systems considered in Section II. In general, when the switching possibilities for basics, involving a transition from one system to another with a different Basic system, are being discussed nonbasics to both systems can be ignored[24].

As an illustration of how a failure to specify whether the commodity that switches is a basic or nonbasic could be misleading, we may refer to Section III of the paper by *Bruno*, *Burmeister* and *Sheshinski* in the Symposium[25]. They consider there first the "canonical model" of *Samuelson*, with one capital good (which is basic) and one consumption good (which is nonbasic) in each production system. While the capital good is different in the two systems the consumption good is the same in both. The consumption good is numeraire and does not use itself in its own production. The authors state correctly, in this case, that there can be at most two switches. (There is only one commodity, the nonbasic consumption good, which is common to the two systems and the two production systems are characterised therefore by the nonbasic being produced by a different method in each. As the nonbasic does not use itself in either of these methods, the maximum number of switching points is only two.) After obtaining the sufficiency conditions for nonreswitching in this

probability that a number of methods should reswitch at the same point (i.e. the same system should return) would be very small. *Levhari*, however, claimed to have established rigorously the *impossibility* of such a reswitching and this claim was certainly wrong.

[24] We may conceive of a peculiar economic situation in which a nation consists of two or more separate economic communities having different customs and therefore, for instance, producing and consuming different kinds of food, etc. They are considered as forming a single statistical aggregate and therefore a single system. The production matrix for such a system would be 'completely decomposable'. (Mathematically, a square matrix A is called 'completely decomposable' when by identical arrangement of rows and columns it can be partitioned into $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ with A_{11} and A_{22} square.) In a completely decomposable system there are no basics as no commodity enters, directly or indirectly, into the production of all commodities in the system. Such a system could be subdivided into the independent economies which are combined to form that system and the commodities in each such economy classified as basics and nonbasics to that economy.

[25] Op. cit., p. 531–538.

simple case they point out the difficulty of generalising these conditions to cases involving more than one capital good in a single method of production. It is here that the error creeps in when they state (p. 536) "The latter fact [the difficulty of so generalising] can be seen by considering a case with one consumption good and two capital goods where the prices are clearly equations of the third degree. Thus in general there may be three switching points". In a production system with two basics and one nonbasic, with the nonbasic as numeraire, prices are not equations of the third degree in the rate of profit unless the nonbasic uses itself as means of production (a condition not present in the canonical model). Further we cannot conclude that the maximum number of switches between two production systems with two basics and one nonbasic in each would be in general three. If following the 'canonical model' we were to assume that the two capital goods (basics) in each system were, in all, four different basics and that the nonbasic did not enter its own production, the two systems have only one commodity, the nonbasic, common to them; there are at most *four* switching points. If the two capital goods in each system were the same two basics and the two systems were characterised by a different method for one of the basics the maximum number of switching points would be at most two. If the two systems had the same two basics, they differed only in the method for the common nonbasic and the nonbasic did not use itself in any of the alternative methods then the maximum number of switches would still be two. The maximum number of switching points would be three when the total number of *different* commodities entering, directly or indirectly, into at least one of the two alternative methods that switch is three[26]. No such condition is specified by the authors and it would seem that they arrived at the conclusion that the number of switches was in general three by counting the commodities in each system.

IV

To sum up:

i) At a switch point the adjacent production systems differ in the method of production for only one of the commodities common to them. The maximum

[26] With two basics and one nonbasic in each system the maximum number of switches would be three when

i) the total number of distinct basics in the two production systems together is three and (a) the two systems are characterised by the use of a different method for a basic common to them, or, (b) the two systems differ only in the method for the nonbasic common to them and neither methods for the nonbasic uses the nonbasic. Or, alternatively when

ii) the total number of distinct basics in the two systems together is two; the two systems differ in the method for the nonbasic and the nonbasic uses itself in at least one of the methods by which it is produced.

number of switching possibilities between two such systems is equal to the number of distinct (i. e. without double counting) commodities entering, directly or indirectly, into the two alternative methods which respectively characterise the two systems. Thus if it is a basic to both systems which has different methods in the two systems, the maximum number of switches would be equal to the total number of distinct basics in the two systems together; if it is a nonbasic which has different methods in the two systems, this maximum number is given by the total number of distinct basics in the two systems *plus* the number of distinct nonbasics entering, directly or indirectly, in at least one of the methods for that nonbasic.

ii) The choice of the value unit does not affect the maximum number of switching possibilities. Nonbasics which require themselves in their own production and which, while not entering, directly or indirectly, into the production of the commodity with alternative methods, do so enter the value unit, give additional formal solutions for switch points. These additional solutions would be ruled out for reasons given on p.417–8 above.

iii) The classification of commodities into basics and nonbasics in a given system uses more of the available information about the system than does the classification of the system as decomposable or indecomposable. The additional information incorporated in the former distinction is essential for the discussion of switching possibilities between two systems.

Appendix

Professor *Newman* in his critique[27] of *Piero Sraffa's* "Production of Commodities by Means of Commodities" raised the issue concerning the necessary and sufficient conditions for all prices to be positive in a production system which includes nonbasics. These conditions (which he states in the article on p. 67) appear to him "to have little economic significance". His conclusion is that the presence of nonbasics in the system "will often not imply a positive price vector". This question of the economic interpretation of these conditions and the treatment of nonbasics were discussed in letters exchanged between *Sraffa* and *Newman*. I sought their permission, which they have kindly given, to publish the letters in full in this Appendix. I here summarise *Newman's* arguments as they appear on p. 66–67 of his article.

Newman first establishes that for a system containing only basics and in which 'labour consumes fixed levels of inputs irrespective of the rate of profit' there is always a solution giving a positive price vector and a positive rate of profit. He then considers a system which includes nonbasics. He gives

[27] *Newman P.*, Production of Commodities by Means of Commodities, Schweizerische Zeitschrift für Volkswirtschaft 1962, p. 58–75.

a simple illustration of a system consisting of only two commodities, iron and corn, where iron (designated commodity 1) is nonbasic and corn (designated commodity 2) is basic. With p_1 and p_2 as prices of iron and corn respectively and r, the uniform rate of profit, the price equations in his example are:

$$\begin{aligned} (1+r)\,0.8p_1 + (1+r)\,0.3p_2 &= p_1 \\ (1+r)\,0.2p_2 &= p_2 \\ 0.2p_1 + \qquad 0.5p_2 &= 1 \end{aligned}$$

In this system if $p_2 \neq 0$ then $(1+r) = 1/a_{22} = 5$ and at that rate of profit $p_1 = -5/4$ and $p_2 = 5/2$. Hence if $p_2 \neq 0$ the solution contains a negative price. If $p_2 = 0$, $p_1 = 5$ and $r = 1/4$. He concludes that "in either case we have a contradiction of *Sraffa's* combined requirements that the system be in a self-replacing state and that profit rate be uniform". *Newman* then states that the necessary and sufficient condition for such a production system having all positive prices is $a_{11} < a_{22}$ (where a_{11} and a_{22} are the iron-iron and corn-corn coefficients respectively). The economic rationale of this condition seems obscure to him. He poses the choice that either we must abandon one of *Sraffa's* assumptions (that there is a uniform rate of profit and that the system is in self-replacing state) or assume that nonbasics do not exist. He favours the course of 'abandoning the nonbasics'. He further adds that "this choice is reinforced by the consideration that the question whether a good is nonbasic is partly a matter of the degree of aggregation in the system". He concludes that "This result, that nonbasics will often not imply a positive price vector, means that the rather heavy emphasis placed on such commodities by *Sraffa* [he exemplifies them by luxury goods] seems misplaced".

The correspondence reproduced below centres on these issues.

Trinity College, Cambridge, England
4th June, 1962.

Dear Professor Newman,

Thank you for sending me your excellent article on my book. I have read it with great interest and I am sure that it will prove illuminating to many who have been puzzled by my work.

There are naturally some points of disagreement. Among these I shall refer only to your criticisms (p. 66–67) of my treatment of non-basic products. Have you not overlooked my Appendix B, to which the reader was referred to by a footnote on p. 28? It seems to say exactly the same thing as you say on p. 66. True, it says it in humdrum economic language, which is no doubt less elegant than mathematics. In this case, however, it has the advantage of making 425 plain the economic circumstances which may give rise to a negative price for a non-basic, and which you find "obscure" (p. 67).

Besides, it makes it obvious how rare (if any) such cases must be in the real world. If, e.g., the ratio of net product to means of production (R) in a basic *system* is 25%, it will be pretty hard to find a *single* commodity (whether basic or not) which requires the using up of more than *four* units of itself in order to produce *five* units of it in a year. I certainly failed to discover any faintly realistic example of this which I could use, and had to invent those "beans".

When you say such instances occur "often" (p. 67) you must have been misled by your own example of a system consisting of a single basic and a single non-basic product – presumably concluding that $a_{11} > a_{22}$ is no less probable than $a_{11} < a_{22}$. In a real system, however, there is not one but a large number of basic products, and the ratio R resulting from the *system* which they form is practically certain to be much smaller than the own ratio of anyone *separate* non-basic (or any of such small groups of interconnected non-basics as may exist).

You find a further ground for attacking the distinction between basics and non-basics in the supposition of its being "partly a matter of the degree of aggregation in the system" (p. 67). Now aggregation is the act of the observer, whilst the distinction is based on a difference in objective properties. I have argued, for instance, that a tax on the price of basics will lower the general rate of profits for a given wage, whereas a similar tax on non-basics will leave the rate of profits unchanged. Surely, to answer this, one must prove the alleged consequence does not follow, instead of drowning the distinction through an appropriate degree of aggregation.

Thank you again for your article. If I may hope for more, it is that you will not really leave your reader to shift for himself in the maze of multiple-product industries.

Yours sincerely,
P. Sraffa

Department of Economics,
The University of Michigan,
Ann Arbor

June 8, 1962.

Dear Mr. Sraffa,

Thank you so much for your letter, and for your kind words concerning my article. It was a relief to learn that I had not badly misinterpreted your ideas,
426 as I feared I might have done.

I can come half-way to meet your criticisms of my treatment of non-basics. I now think that there is some economic meaning to Gantmacher's conditions (p. 92) for the positivity of prices. Let us designate the reducible system $Ap = cp$ by

$$\begin{bmatrix} A_B & 0 \\ A_{BN} & A_N \end{bmatrix} \begin{bmatrix} P_B \\ P_N \end{bmatrix} = c \begin{bmatrix} P_B \\ P_N \end{bmatrix}$$

where A_B and A_N are the square 'internal' coefficient matrices for basic and non-basic goods respectively, A_{BN} is the (in general nonsquare) matrix of coefficients of basic goods used in non-basic good manufacture, P_B and P_N are sub-vectors of the respective prices, and c is A's dominant latent root. Then we can consider A_B and A_N as themselves matrices like A, with dominant latent roots c_B and c_N respectively, and associated eq. 'rates of profit' r_B and r_N.

Then Gantmacher's necessary and sufficient condition for positivity of P_B and P_N may be expressed as $r_B < r_N$, i. e., the rate of profit in the basic system must be strictly less than the rate of profit of the 'internal' non-basic system. This seems to have economic meaning, though I am not sure about its significance. I confess that it does not seem to me to be obvious that we will usually have $r_B < r_N$, but I am open to argument. It seems to me that more empirical considerations would have to be brought in.

I would not have brought in the point about aggregation if I had not already made the earlier, and I think stronger, point. I do wonder a little about your mention of 'objective properties'. All we ever have is what we observe, or more strictly, what we classify. I personally find it difficult to think in terms of industries when considering production, and think more naturally of processes. For this reason, I think further discussion of this point would not be useful, since I imagine that we would both agree that the Part II analysis of processes is a considerable step forward. I have not thought about the role of aggregation in the latter context.

Your invitation to work on Part II of the book is very enticing. My free time is rather limited just now, and I suspect it will take much harder work than Part I. But I might steal time to work at it.

With best wishes.

Yours sincerely,
Peter Newman

Trinity College, Cambridge
19th June, 1962.

Dear Professor Newman,

Thank you so much for your letter.

I am, of course, delighted, and grateful, that you can come half-way to meet me on the subject of non-basics, and I only regret to be unable to move the other half: I cannot yield an inch on this point! 427

You speak of a non-basic system and proceed to compare it with the basic system: I say that there is no such thing as a non-basic system. You also refer to "the rate of profit of the internal non-basic system": again, I say there is no such thing.

It is in the nature (or, if you wish, the definition) of basic goods to be interconnected and form a system. It is, on the other hand, the peculiarity of non-basics to be unconnected with one another, and they are incapable of forming an independent system. At best, each of them can be formally treated as constituting a separate single-commodity system, with its own rate of profits: this rate (for each separate non-basic) can be compared with the rate of the basic system. It is *a' priori* extremely unlikely that any individual rate will be smaller than that of the basic system, composed, as the latter is, of many products, all used directly or indirectly in one another's production. It has not been possible to find a reasonable case in reality in which the rate is smaller (and this is not a minute, hidden property that requires elaborate investigation for spotting it).

If I may go over the ground again. The immense majority of non-basics are not used in production, not even in their own production: so they do not even form individual systems. Some (mainly animals and plants) are used each in its own reproduction, and form individual systems. A few may be linked with one or two others, because of mixing, or cross-breeding, or if the length of gestation brings out the egg-hen dicotomy. And that is all.

The third class, which is the least numerous and may just be worth mentioning for the sake of completeness, is the source of all the trouble.

With many good wishes.

Yours sincerely,
P. Sraffa

Summary

On the Maximum Number of Switches Between Two Production Systems

This paper discusses, adopting Piero Sraffa's classification of commodities into 'basics' and 'nonbasics', the question of the maximum number of switches between two production systems in the most general case where the two systems are characterised by 'basics' not all common between them and where they may include 'nonbasics'. This maximum number is given by the number of distinct (i. e. without double counting) commodities entering, directly or indirectly, into the two alternative methods that characterise the two systems – the two systems adjacent at a switch point differing in the production method for only one of the commodities common between them. The paper brings out the particular advantage of the 'basic-nonbasic' distinction for this discussion. In the Appendix are published letters exchanged between Piero Sraffa and Peter Newman on the role of 'nonbasics'.

Zusammenfassung

Über die maximale Anzahl von Wechseln zwischen zwei Produktionssystemen

Dieser Aufsatz diskutiert, ausgehend von Piero Sraffas Klassifikation der Güter in « basics » und « nonbasics », die Frage der maximalen Anzahl der Wechsel zwischen zwei Produktionssystemen im allgemeinsten Fall. Hier zeichnen sich diese Systeme durch verschiedene « basics » aus, wobei aber beide Systeme « nonbasics » enthalten mögen. Diese maximale Anzahl wird durch die Zahl der verschiedenen Güter (d.h. Doppelzählungen sind ausgeschlossen) bestimmt, welche direkt oder indirekt in die zwei alternativen Methoden eingehen, welche die beiden Produktionssysteme charakterisieren. Dabei besitzen beide Systeme an einem Wechselpunkt eine grosse Ähnlichkeit, da sie nur einen Unterschied in der Produktionsmethode bezüglich eines der gemeinsamen Güter aufweisen.

Der Aufsatz zeigt den besonderen Vorteil der Unterscheidung in « basics » und nonbasics » für diese Diskussion. Im Anhang wird der Briefwechsel zwischen Piero Sraffa und Peter Newman über die Rolle der « nonbasics » publiziert.

Résumé

Le nombre maximum de changements entre deux systèmes de production

En adoptant la classification de Piero Sraffa des biens en « basics » et « nonbasics », l'article discute la question du nombre maximum de changements entre deux systèmes de production dans le cas le plus général où les deux systèmes sont caractérisés par des « basics » étant entre eux tout à fait différents et où les systèmes pourraient comprendre des « nonbasics ». Le nombre maximum est déterminé par la quantité de biens différents (c'est-à-dire sans dénombrement double) entrant directement ou indirectement dans les deux méthodes alternatives qui caractérisent les deux systèmes. Les deux systèmes, similaires à un point de changement, ne diffèrent dans la méthode de production qu'en un des biens communs entre eux. L'article montre l'avantage particulier de la distinction entre « basics » et « nonbasics ». Dans l'appendice, on publie des lettres échangées entre Piero Sraffa et Peter Newman sur le rôle des « nonbasics ».

[12]

Contributions to Political Economy (1985) **4**, 61–77

DUALITY AND POSITIVE PROFITS*

HARVEY GRAM

Queens College, City University of New York

INTRODUCTION

The principle of duality is the basis for many short-cuts in economic theory. The properties of expenditure functions and cost functions, for example, are often used to obtain results which are much more difficult to establish using utility functions and production functions. This, in turn, gives firm theoretical foundation to much empirical work. It is somewhat surprising that these uses of duality have only recently become popular. Although the work of Hotelling (1932), Roy (1942), Houthakker (1951–52) and Shephard (1953, 1970) has long been recognised, it is the systematic development and generalisation of these earlier contributions by McFadden (1978) and others which has focused attention of the analytical power of the duality principle.[1]

Duality arguments are commonly used in the theory of international trade where general equilibrium models have always played a prominent role. In the development of this theory, Samuelson's 'Prices of factors and goods in general equilibrium' (Samuelson, 1953–54) has been strikingly influential. After more than 30 years it continues to be cited by trade theorists more frequently than any other paper.[2]

Samuelson expressed duality in terms of a set of 'reciprocity relations' derived from his national income or national product function. The purpose of this paper is

*Various versions of this paper have been given at seminars at Carleton University, the University of Denver, and Queens College of the City University of New York. I would like to acknowledge the many helpful comments made by my colleagues on these occasions. I am particularly grateful to John Eatwell, Peter Newman and Ian Steedman for their careful reading of the paper in its penultimate version, but I am responsible for remaining errors. This research was supported (in part) by grant number 13466 from the PSC–CUNY Research Award Program of the City University of New York.

[1]The recent survey by Diewert (1982) includes a discussion of the historical development of duality theory.

[2]See, for example, the list of references to almost any paper on the 'pure theory' of trade in the *Journal of International Economics*. See also the prominent place given to Samuelson's paper by Jones (1984) and Ethier (1984) in their respective contributions to Volume I of the recent *Handbook of International Economics*.

0277–5921/85/010061 + 17 $03.00/0

to reformulate, in the context of a model with positive profits, the particular reciprocity relation most frequently used in trade theory; namely, the set of equations linking the effects of commodity price changes on income distribution to the effects of factor supply changes on the composition of output. Samuelson stressed unequivocally that 'inter-industry circular whirlpools do not in any way affect the conclusions' (Samuelson, 1953–54, p. 19) concerning these equations. On the contrary, our argument shows that 'whirlpools' cannot be ignored except when profit rate and growth rate are equal.[1] Without this assumption, duality no longer provides the theoretical basis for many of the arguments frequently made in trade theory. We show this by establishing a new form for Samuelson's reciprocity relation.

DUALITY THEORY

The duality principle under discussion follows from Shephard's lemma. This famous result states that, where it exists, the partial derivative of a cost function with respect to the price of an input is the unique cost-minimising requirement of that input in the production of the output for which the cost function is defined. This is a constant per unit requirement where there are constant returns to scale, an assumption we will make throughout our argument.[2] Shephard's lemma is familiar from neoclassical price theory. A pair of cost functions (average or total) are drawn on the assumption that one function takes all inputs but one as given and the other allows more than this one input to vary. At some level of output the functions are tangent; at all other levels of output, the function allowing more inputs to vary exhibits a lower cost. At the point of tangency, where the freedom to vary more than one input is of no consequence, the rate of change of cost with respect to a change in the price of some variable input is independent of any change in the production process. We now proceed to make use of this result.

In our model, m final commodities (non-basic in the sense of Sraffa) are produced using q non-produced factors (in fixed supply) and n circulating capital goods, the latter being produced with the same factors and themselves as inputs. The vector of factor supplies, $\mathbf{v}$, is a parameter; the vector of final outputs, $\mathbf{x}$, is determined by the

[1] Because Samuelson assumed a stationary state when he considered the effects of produced inputs, and because a positive rate of growth would only complicate our discussion, we assume throughout that the growth rate is zero. Positive profits therefore imply a profit rate in excess of the growth rate. See, however, the later footnote on this topic.

[2] It is interesting to note that the only doubts expressed by Samuelson about the relevance of his analysis are in connection with hypotheses concerning non-increasing returns to scale and the additivity, without 'external' inter-actions, of production processes. Thus, 'there is considerable empirical evidence, in connection with technology and the breakdown of perfect competition, that in large realms of economic life these are poor hypotheses to make' (Samuelson, 1953–54, p. 2). There has also been a lively debate concerning the significance of constant returns to scale in the analysis of a Sraffa system. See, for example, Burmeister (1977) and Eatwell (1977). We do not enter into this discussion because of our limited purpose which is simply to consider the implications of a positive rate of profit in a model which is otherwise identical to Samuelson's model.

composition of demand; and the vector of produced inputs, **y**, is appropriate to **x** in stationary equilibrium. Thus, **y** is appropriate to maintaining a constant flow of final output, an assumption implicit in the last part of Samuelson (1953–54, pp. 17–20). The profit rate, r, is exogenously given; the vector of market clearing commodity prices, π, implies the vector of cost-minimising factor prices, **w**; and the vector of capital good prices, **p**, is consistent with the same **w** and r. Formally:

$$\beta\mathbf{x}+b\mathbf{y}=\mathbf{v}, \tag{1}$$

$$\alpha\mathbf{x}+a\mathbf{y}=\mathbf{y}, \tag{2}$$

$$\pi'=\mathbf{w}'\beta+\mathbf{p}'(1+r)\alpha, \tag{3}$$

$$\mathbf{p}'=\mathbf{w}'b+\mathbf{p}'(1+r)a, \tag{4}$$

where a prime indicates a row vector, all other vectors being column vectors. We have followed Hicks (1965), using Greek letters β and α for the $q\times m$ and $n\times m$ matrices of factor requirements and capital good requirements per unit output of final commodities, and Roman letters b and a for the corresponding $q\times n$ and $n\times n$ matrices in the production of circulating capital goods. An 'interior' solution is assumed to exist, which is to say that the resource constraints in (1) can be satisfied as equations at a point where all outputs are positive; and commodity prices are such that all factor prices are positive. Finally, the prices of non-basics are such that the implied factor prices (which depend on the rate of profit, as we will show presently) generate factor incomes which (together with profit incomes) result in a composition of demand just equal to the composition of final output.[1] For this to be possible, the value of net output must be equal to the value of factor payments plus net profits. Adding (1), pre-multiplied by $\mathbf{w}'$, and (2), pre-multiplied by $(1+r)\mathbf{p}'$, yields an expression for gross income. Adding (3), post-multiplied by **x**, and (4), post-multiplied by **y**, yields an expression for gross output. Subtracting depreciation (the value of circulating capital, $K=\mathbf{p}'\mathbf{y}$) from each gives the required net product, net income equation:

$$Y=\pi'\mathbf{x}=\mathbf{w}'\mathbf{v}+r\mathbf{p}'\mathbf{y}. \tag{5}$$

As Appendix A shows in somewhat greater detail, Shephard's lemma allows us to write the total differential of the price equations, using (A.2) and (A.3), as

$$(\mathrm{d}\pi')\mathbf{x}=(\mathrm{d}\mathbf{w}')\mathbf{v}+r(\mathrm{d}\mathbf{p}')\mathbf{y}. \tag{6}$$

[1] Hahn (1982) points out that cost of production prices in a Sraffa system are simultaneously supply and demand prices. But, in his model, there is only one non-produced factor. With more than one such factor (but still assuming that the supply of produced inputs is determined by the composition of final output) this is no longer necessarily true, and so we must add the stipulation that the supply of each non-basic, at cost of production prices, is equal to the demand for it.

It immediately follows from the total differential of net product in (5) that

$$\boldsymbol{\pi}'(\mathbf{dx}) = \mathbf{w}'(\mathbf{dv}) + r\mathbf{p}'(\mathbf{dy}). \tag{7}$$

These are new results. They show that neither the quantity weighted sum of price changes, $(\mathrm{d}\boldsymbol{\pi}')\mathbf{x} - (\mathbf{dw}')\mathbf{v}$, holding factor supplies constant, not the price weighted sum of quantity changes, $\boldsymbol{\pi}'(\mathbf{dx}) - \mathbf{w}'(\mathbf{dv})$, holding prices constant, is zero unless the profit rate is zero.[1] It is this fact which has important implications for the dual relationships between price changes and quantity changes. To show this, take the partial derivative of Y with respect to π_j:

$$\partial Y/\partial\pi_j = x_j + \sum_k \pi_k \partial x_k/\partial\pi_j = x_j + r\sum_k p_k \partial y_k/\partial\pi_j \tag{8}$$

using (7) with $\mathbf{dv} = 0$; and the partial derivative of Y with respect to v_i:

$$\begin{aligned}\partial Y/\partial v_i &= w_i + \sum_h v_h \partial w_h/\partial v_i + r(\sum_h y_h \partial p_h/\partial v_i + \sum_h p_h \partial y_h/\partial v_i)\\ &= w_i + r\sum_h p_h \partial y_h/\partial v_i,\end{aligned} \tag{9}$$

using (6) with $\mathrm{d}\boldsymbol{\pi} = 0$. Then, set equal the symmetric second order cross partial derivatives of Y with respect to both π_j and v_i, using (8) and (9) to obtain[2]

$$\partial w_i/\partial\pi_j = \partial x_j/\partial v_i + r\Omega_{ij}, \tag{10}$$

where

$$\Omega_{ij} = \sum_k (\partial p_k/\partial v_i)\,(\partial y_k/\partial\pi_j) - \sum_h (\partial p_h(\partial\pi_j)\,(\partial y_h/\partial v_i).$$

Equation (10) reduces to Samuelson's 'reciprocity relation' when the profit rate is zero; namely, that the effect of a difference in the price of final good j on the equilibrium value of factor service i (holding factor supplies constant) is exactly equal to the effect of a difference in the supply of factor i on the equilibrium output of commodity j (holding commodity prices constant). With positive profits these effects are not equal and may even differ in sign (see Appendix B). Samuelson's

[1] For a zero rate of profit, equation (7) states that a hyperplane tangent to the production possibilities surface has a gradient vector given by relative price ratios, i.e. marginal rates of transformation are measured by relative prices. In two dimensions, (7) becomes $\mathrm{d}x_2/\mathrm{d}x_1 = -\pi_1/\pi_2$ when factor supplies are given ($\mathrm{dv} = 0$) and the profit rate is zero. When the profit rate is positive;

$$\mathrm{d}x_2/\mathrm{d}x_1 = -[\pi_1 - rp_1(\mathrm{d}y_1/\mathrm{d}x_1)]/[\pi_2 - rp_2(\mathrm{d}y_2/\mathrm{d}x_2)].$$

The neoclassical equality between marginal rates of transformation in production and marginal rates of substitution in consumption (assuming that consumers equate the latter to relative price ratios) is no longer a feature of competitive equilibrium when r is positive. For further discussion of the inequality between marginal rates of transformation and relative prices, see Burmeister (1975).

[2] Shephard's Lemma and equations such as (6) and (7) also characterise models with joint production, as shown by Jones and Scheinkman (1977). Equation (10) can also be derived for such models.

claim that the 'inter-industry circular whirlpools' make no difference to his results depends therefore on the unstated assumption that the profit rate is zero in a stationary state. The more general condition is given in equation (10).[1]

INTERPRETATION

An interpretation of (10) based on an analysis of the final term, which incorporates the effects of factor supply changes and commodity price changes on the value of the stock of capital, would be needlessly complex. Consider instead a reduced form of the quantity and price equations:

$$B\mathbf{x} = \mathbf{v}, \tag{11}$$

$$\pi' = \mathbf{w}'C, \tag{12}$$

where

$$B = \beta + b[I - a]^{-1}a,$$

$$C = \beta + b[I - (1+r)a]^{-1}(1+r)a,$$

of which typical elements are

$$B_{ij} = \beta_{ij} + \sum_k b_{ik}a_{kj} + \sum_k \sum_h b_{ik}a_{kh}a_{hj} + \ldots,$$

$$C_{ij} = \beta_{ij} + (1+r)\sum_k b_{ik}a_{kj} + (1+r)^2 \sum_k \sum_h b_{ij}a_{kh}a_{hj} + \ldots.$$

If the corresponding elements of B and C were set equal by imposing a zero rate of profit, there would be little difference between the analysis of a model with

[1] Allowing for proportional growth is not difficult, but begs the question as to why the supply of primary factor sevices should increase at the same rate as the rate of accumulation of reproducible stocks of capital goods. Formally, simply multiply a and a in (2) by $(1+g)$ and define net income as

$$\begin{aligned} Y &= \pi'\mathbf{x} + gK = \mathbf{w}'\mathbf{v} + rK \\ &= \pi'\mathbf{x} + (1+g)^{-1}g\mathbf{p}'\mathbf{y} = \mathbf{w}'\mathbf{v} + (1+g)^{-1}r\mathbf{p}'\mathbf{y}, \end{aligned}$$

using $K = \mathbf{p}'(a\mathbf{x} + a\mathbf{y})$ and $\mathbf{p}'\mathbf{y} = (1+g)K$. Introducing Z for the value of consummable output,

$$Z = \pi'\mathbf{x} = \mathbf{w}'\mathbf{v} + [(r-g)/(1+g)]\mathbf{p}'\mathbf{y},$$

it is now a simple matter to derive equation (10) using Z in place of Y. In the final result, r is replaced by $(r-g)/(1+g)$. This reduces to r when g is zero and also shows that when growth is positive, it is equality between profit rate and growth rate which is required for Samuelson's 'reciprocity relationship'. It should be understood, however, that this is a fluke case which gives no support for the systematic relationships between prices and quantities characteristic of neoclassical theory. Thus, in Appendix B, the direct and indirect factor requirements in the B matrix remain as we have defined them and can vary in unorthodox ways when the profit rate is positive, even though the growth rate may be equal to the profit rate. Note that it would be inappropriate to redefine B as equal to C with g replacing r, for one has then simply forced an interpretation on the coefficients in the quantity equations guaranteed to generate neoclassical results.

produced inputs and one without. Indeed, this is precisely the basis for Samuelson's argument (Samuelson, 1953–54, pp. 17–20). But when r is positive, a mark-up equal to $(1+r)^t$ is applied, in the price equations only, to the indirect factor services used 't periods ago' to produce the capital goods (and the capital goods used to produce those capital goods, etc.) which are needed in the production of final commodities. Direct and indirect factor costs, determined by the cost-minimising C matrix, will not (except by a fluke) be proportional to the generally smaller direct and indirect factor requirements of the corresponding B matrix. In the simplest case, where there is only one technique, a positive rate of profit will cause differences and possibly reversals of relative factor intensity, comparing price equations with quantity equations (see Appendix B). Moreover, where substitution among inputs is allowed, cost minimisation at a given rate of profit may be associated with upward sloping 'demand' curves for factors (also shown in Appendix B).

Whatever the peculiarities revealed by particular examples, the general result of our analysis is revealed in a straightforward way by equations (11) and (12). A positive rate of profit means that, in the calculation of total factor costs, indirect factor requirements must be compounded by an appropriate profit factor. Finding the C matrix which minimises factor cost simultaneously determines a B matrix, but there is no reason to expect the relationship between factor prices and the elements of B to exhibit any of the systemtic properties of neoclassical theory.

DUALITY IN TRADE THEORY

Samuelson's 'reciprocity relation' plays a central role in recent statements of neoclassical trade theory (Jones, 1984; Ethier, 1984). It allows a tight link to be drawn between two famous results: the Stolper–Samuelson theorem and the Rybczynski theorem. These may be stated as follows. If we assume non-joint production, every commodity price change, measured in percentage terms, is a weighted average of the associated factor price changes, also measured in percentage terms. The weights are sectoral factor income shares. To show this, differentiate (12), assuming all factor prices positive before and after a change in commodity prices,[1] to obtain[2]

$$\hat{\pi}_j=\sum_i \theta_{ij}(r)\hat{w}_i, \qquad j=1,\ldots,m, \tag{13}$$

where a circumflex indicates a proportional differential change (as in $\hat{\pi}_j=\mathrm{d}\pi_j/\pi_j$) and $\theta_{ij}(r)=w_iC_{ij}(r)/\pi_j$ is the share of compounded factor i costs per unit value of commodity j. Because these shares add up to unity in each sector, it follows from (13) that some $\hat{w}_i$ must be larger than the largest $\hat{\pi}_j$ and some other $\hat{w}_i$ must be smaller than the smallest $\hat{\pi}_j$ (and therefore negative if some commodity price is constant or falling). Thus, for non-joint production technologies, there is a clear gainer and a

[1] For a more general treatment, see Ethier (1984, pp. 165–166).

[2] This result depends on the absence of joint production, whereas the 'reciprocity relation' does not.

clear loser associated with any given change in the commodity price vector. In a two-factor, two-commodity model, the gaining factor is the one for which relative factor cost is greatest in the production of the commodity whose relative price has risen; the other factor price falls in terms of both goods. This was the basis for the famous paper by Stolper and Samuelson (1941) concerning the effects of tariffs on the distribution of income. Jones (1965) has described the Stolper–Samuelson theorem as an example of the sort of 'magnification effect' which characterises non-joint production models: a change in relative prices of outputs causes a magnified reaction on the relative prices of inputs.

A similar argument concerning quantities requires the additional assumption that the number of final commodities is not less than the number of primary factors.[1] In that event, constant commodity prices imply constant factor prices because the C matrix in (12) is square or can be made square by dropping commodities, and is therefore invertible (assuming no linear dependencies among its columns). The technique matrix is therefore unchanged in the face of factor supply changes at constant commodity prices. Assuming all outputs positive before and after a change in factor supplies,[2] the quantity equations in (11) yield

$$\sum_j \lambda_{ij}\hat{x}_j = \hat{v}_i, \qquad \mathrm{i}=1,\ldots,q, \tag{14}$$

where $\lambda_{ij}=B_{ij}x_j/v_i$ is the fraction of total employment of factor i accounted for directly and indirectly by the production of final good j. Because these shares add up to one (across sectors rather than within sectors, as in the case of income shares), every proportional factor supply change is bounded by some larger and some smaller proportional output change. One sector expands relatively to all quantities and one sector contracts relative to all quantities (shrinking absolutely if at least one factor supply is constant). This second 'magnification' effect, made famous by Rybczynski (1955) is relevant to a small open economy facing given prices and experiencing uneven factor growth. In the two-factor, two-commodity case, the sector that expands relatively to the increase in the faster growing factor produces the commodity that uses that factor relatively intensively. The other sector contracts relatively (and absolutely if one factor supply is constant or falling).[3]

The problem faced in generalising the Stolper–Samuelson and Rybczynski theorems is somehow to identify the gaining and losing factors and the expanding and contracting sectors. Restrictions on technology sufficient for this purpose have not provided useful results because the restrictions generally imply consistent

[1]Ethier (1984, pp. 178–181) suggests that an alternative assumption — that the number of international markets is not less than the number of primary factors — permits the generalisation of many results which seem at first to depend upon an arbitrary facet of technology, namely, $m \geq q$.

[2]Again, Ethier (1984, pp. 169–196) considers a more general case.

[3]This result turns critically on the dimensions of the model. With three factors and two final products, for example, it need not hold. See Jones (1971).

aggregation to a two-factor, two-commodity specification.[1] Recent arguments, seeking to establish weaker generalisations than those originally attempted, rely on Samuelson's 'reciprocity relation'. The following are the main two propositions. Every commodity is an 'enemy' to some factor price, and every sector has an 'enemy' in the quantity of some factor supply. Every commodity is a 'friend' to some factor price, and every sector has a (qualified) 'friend' in the quantity of some factor supply. These are in the nature of existence arguments. What duality theory seeks to accomplish is to identify the price and quantity 'friends' and 'enemies' indirectly. What we now show is simply that the usual arguments, in fact, provide no definite results when the profit rate is positive.

The first part of each of the above two statements follows from (13). Change *one* commodity price and there will be some factor that loses unambiguously and another that gains unambiguously. Suppose that these factors have been identified from an econometric study of cost functions. Rewrite 'reciprocity relation' (10) as

$$(\hat{w}_i/\hat{\pi}_j) = (\varphi_j/\psi_i)\,(\hat{x}_j/\hat{v}_i) + r\hat{\Omega}_{ij}/\psi_i, \tag{15}$$

where $\psi_i = w_i v_i/Y$ and $\varphi_j = \pi_j x_j/Y$ are economy-wide factor and product shares, and[2]

$$\hat{\Omega}_{ij} = \sum_h (rp_h y_h/Y)\,[(\hat{p}_h/\hat{v}_i)\,(\hat{y}_h/\hat{\pi}_j) - (\hat{p}_h/\hat{\pi}_j)\,(\hat{y}_h/\hat{v}_i)].$$

It follows from (15) that if i identifies the unambiguously losing factor, both sides of the equation are negative. Therefore, if $r=0$, this same factor must be the 'quantity enemy' of sector j. But even this weak proposition — weak because it assumes that we know which factor loses from an increase in the price of product j — need not hold when r is positive because $\hat{\Omega}_{ij}$ can be negative. It is no use to *assume* $r=0$ when it is not since the factor that loses from an increase in π_j would generally be a different one for different values of r.

On the other hand, suppose in (15) that i identifies the unambiguously gaining factor. Then, both sides of the equation exceed unity and so, for $r=0$, sector j will expand relatively to an increase in the supply of factor i, provided that the share of factor i in total income is large enough relative to the share of sector j in total product. In other words, φ_j/ψ_i must be small enough so that $\hat{x}_j\hat{v}_i$ can exceed unit when $\hat{w}_i/\hat{\pi}_j$ does. In this sense a factor must be large enough relative to a sector in order to qualify as a quantity 'friend'. But, even this weak proposition need not hold when r is positive because $\hat{\Omega}_{ij}$ can be positive.[3] In short, the attempt to identify quantity

[1] See the discussion in Ethier (1984, pp. 149–161).

[2] Note that $\Sigma_j\varphi_j = \Sigma_i\psi_i + r\Sigma_h\gamma_h$, where $\gamma_h = rp_h y_h/Y$ is the share of total profit income accruing to the owners of capital good h.

[3] It should be noted that there is another definition of 'friendship' discussed in the trade literature; namely, that the share of the value of total output accounted for by sector j should be higher after an increase in the supply of factor i (because of a reduction in the net value of the aggregate of all other outputs). When $r=0$, the factor which gains unambiguously from a rise in the price of commodity j is also the factor which is a quantity friend to sector j in this weaker sense, regardless of the value of the ratio of shares, φ_j/ψ_i. See Jones (1979, pp. 122–123).

'friends' and 'enemies' indirectly from information about price 'friends' and 'enemies' by means of the duality principle fails when the profit rate is positive.

Other propositions, such as 'Every factor is a quantity enemy to some sector and a quantity friend to some other sector', follow from (14) and therefore depend on a dimensionality restriction; namely, that the number of factors does not exceed the number of goods. But, even supposing this restriction to be satisfied, it does not follow, as trade theorists have claimed, that price 'enemies' and (qualified) price 'friends' can be identified from information about quantity 'enemies' and 'friends'. Again, the argument is based on (15). A negative $\hat{x}_j/\hat{v}_i$ means that factor i is the quantity 'enemy' of sector j, but when $r\hat{\Omega}_{ij}$ is positive this does not mean that commodity j is thereby identified as the price 'enemy' of factor i. A value for $\hat{x}_j/\hat{v}_i$ in excess of unity means that factor i is the quantity 'friend' of sector j, but when $r\hat{\Omega}_{ij}$ is negative this does not mean that $(\psi_i/\varphi_j)\,(\hat{w}_i/\hat{\pi}_j)$ is necessarily also greater than unity, which is the criterion for identifying commodity j as the (qualified) price 'friend' of factor i (qualified in the sense that the share of product j in total output must be 'large enough' relative to the share of factor i in total income so that $\hat{w}_i/\hat{\pi}_j$ is greater than unity).

The upshot of this discussion is straightforward. In the presence of positive profits, the tight link between Stolper–Samuelson and Rybczynski theorems is broken because of the difference between the coefficients of quantity and price equations. It follows that any attempt to generalise these theorems by an appeal to duality is bound to fail if it does not take account of the effects of factor supply changes and commodity price changes on the composition and value of the stock of produced means of production (as represented in our model by the term $r\hat{\Omega}_{ij}$).

There are other problems in trade theory which have also been analysed using Samuelson's 'reciprocity relation'. In the literature on international factor mobility, for example, it has been argued that in a two-factor, two-commodity world the effect of a 'capital inflow' on the domestic output of an imported good is exactly equal to the effect of a change in the terms of trade (the relative price of the imported good) on the return to capital which must be paid to foreigners (Jones, 1967, p. 8). In the models in which such relationships are used, 'capital' is treated like a primary factor, fixed in total supply but mobile internationally.[1] Recognising that capital goods are produced inputs would alter the entire structure of the models under discussion giving a different form to the questions posed. The analytical role of the duality principle and of Samuelson's 'reciprocity relation', in particular, would therefore have to be completely reconsidered.

[1]This branch of trade theory is concerned with the optimal location of 'capital', assuming the other factor, labour, to be immobile. Since the 'capital' is indistinguishable from 'land', however, the theory can be interpreted as addressing itself to the question of the optimal location (from the point of view of one country) of the boundary between it and the rest of the world (assuming that labour does not move with the land when the boundary is changed). It is recognised that this approach to international capital movements which simply ignores the endogenous nature of a reproducible stock of produced inputs, 'rules out many of the interesting phenomena associated with models of growth' (Jones, 1967, p. 37).

CONCLUSION

The model we have used as the basis for a re-examination of Samuelson's 'reciprocity relation' is a hybrid. The vector of non-reproducible factors of production is a parameter, but the vector of reproducible capital goods is a variable determined by the composition of final output. And, with respect to prices, although the pattern of relative factor prices is determined endogenously by the vector of commodity prices, the overall level of factor prices is fixed by the exogenously given rate of profit.[1] Such an approach is, of course, an anathema to neoclassical theorists (see Hahn, 1982) for whom any exogenous distribution parameter immediately signals a methodological error.[2] Indeed, Samuelson is at pains to point out the almost certain error implicit in attempting to argue from factor prices to commodity prices, rather than the other way around: 'you simply *cannot* specify all real w's arbitrarily: the result may simply not be feasible or may be grossly conservative' (Samuelson, 1953–54, p. 4). Of course, this would be true of factor prices in our model, too, given *any* profit rate between zero and the maximum possible r.[3] And, certainly, Samuelson may be interpreted as taking the profit rate as exogenous *and* equal to zero. But the matter cannot be left there for, as Samuelson remarks in his discussion of intermediate inputs, 'we assume statical conditions so that none of the problems of capital and interest arise' (Samuelson, 1953–54, p. 17). The simplest answer to this assertion is that capital theoretic problems cannot be avoided whenever the stock of intermediate inputs is endogenously determined as in a Leontief-type model of the kind used by Samuelson. The alternative neoclassical approach is to treat all inputs (whether produced or not) as arbitrarily 'given' factors. One would not then expect to find equilibrium rental rates bearing any systematic relationship to equilibrium prices of reproducible inputs (such a relationship being, in our model, one of proportionality where the factor of proportionality is $1+r$). The question then arises as to what sort of equilibrium such a model, with arbitrarily given stocks of produced and non-produced inputs, is intended to describe. The neoclassical answer is to extend the concept of equilibrium to embrace the future as well as the present. Rental rates, *inclusive* of expected capital gains and losses (which are always realised in equilibrium), do then bear such a relationship to the prices of produced inputs that a uniform rate of return is realised. Neoclassical theorists

[1] For given commodity prices, a lower rate of profit would not necessarily entail an equal proportionate rise in all factor prices. However, since no element of the C matrix increases when the profit rate is lower, it follows that no price constraint in (12) shifts in towards the origin. Thus, it may be said that no factor price falls when r falls and, in general, some and possibly all such prices increase. In this sense, the level of factor rewards varies inversely with r.

[2] Of course, this is somewhat ironic since all neoclassical models take the distribution of ownership of 'resources' as a datum, suitably restricted to ensure that zero incomes are ruled out (Arrow and Hahn, 1971, p. 117). The latter is necessary, not to deal in any way with the problem of subsistence, but rather to ensure that demand correspondences are suitably continuous.

[3] As is well known, the maximum r in our model is the reciprocal of the dominant characteristic root of the matrix, a, minus 1. See Pasinetti (1977).

make no very great claims for this theory because 'on the manner in which such an equilibrium is supposed to come about, neoclassical theory is highly unsatisfactory' (Hahn, 1982, p. 373). For Joan Robinson, of course, this admission was fatal. There is, however, a further objection. The intertemporal equilibrium models put forward by Hahn and others as an alternative to Sraffa-type models (of which ours is a variant) can never be *out* of equilibrium (cf. Garegnani, 1976). There is no long period position implicit in the 'present' situation and relative to which a short period analysis of deviations from equilibrium could be analysed. Moreover, any event (including purely random events) which alters the parameters of the supply and demand equilibrium simultaneously alters the whole future 'perfect foresight' equilibrium of the economy. Any yet, such events are themselves unpredictable.

We conclude from this that any objections to our argument based on the claim that the rate of profit should be explained in terms of a neoclassical intertemporal equilibrium theory is irrelevant. Such a theory, however unsatisfactory even to its own inventors, provides no basis for the comparison of long period positions which are central to the kinds of comparative equilibrium arguments in which the duality principle has provided such a powerful analytical tool. Our reconsideration of this principle in the presence of positive profits shows how a frequently used 'reciprocity relation' characteristic of a wide class of neoclassical general equilibrium models, in fact, provides no basis for the systematic relationship between price changes and quantity changes which is attributed to it.

REFERENCES

Arrow, K. J. and Hahn, F. H. (1971). *General Competitive Analysis,* San Francisco, Holden-Day.

Burmeister, E. (1975). Many primary factors in non-joint production economies, *Economic Record,* December.

Burmeister, E. (1977), The irrelevance of Sraffa's analysis without constant returns to scale, *Journal of Economic Literature,* March.

Diewert, W. E. (1982). Duality approaches to microeconomic theory, in Arrow, K. J. and Intriligator, M. D. (eds), *Handbook of Mathematical Economics,* Vol. II, Amsterdam, North-Holland.

Eatwell, J. (1977). The irrelevance of returns to scale in Sraffa's analysis. *Journal of Economic Literature,* March.

Ethier, W. J. (1984). Higher dimensional issues in trade theory, in Jones, R. W. and Kenen, P. B. (eds), *Handbook of International Economics,* Vol. I, Amsterdam, North-Holland.

Garegnani, P. (1976). On a change in the notion of equilibrium in recent work on value and distribution, in Brown, M., Sato, K. and Zarembka, P. (eds), *Essays in Modern Capital Theory,* Amsterdam, North-Holland.

Hahn, F. H. (1982). The neo-Ricardians, *Cambridge Journal of Economics,* December.

Hicks, J. (1965). *Capital and Growth,* London, Oxford University Press.

Hotelling, H. (1932). Edgeworth's taxation paradox and the nature of demand and supply functions, *Journal of Political Economy,* October.

Houthakker, H. S. (1951–52). Compensated changes in quantities and qualities consumed. *Review of Economic Studies,* June (No. 50).

Jones, R. W. (1965). The structure of simple general equilibrium models, *Journal of Political Economy,* December.

JONES, R. W. (1967). International capital movements and the theory of tariffs and trade, *Quarterly Journal of Economics,* February.
JONES, R. W. (1971). A three-factor model in theory, trade, and history, in BHAGWATI, J. N., JONES, R. W., MUNDELL, R. A., and VANEK, J. (eds), *Trade, Balance of Payments and Growth,* Amsterdam, North-Holland.
JONES, R. W. (1979). *International Trade: Essays in Theory,* Amsterdam, North-Holland.
JONES, R. W. (1984). The positive theory of international trade, in JONES, R. W. and KENEN, P. B. (eds), *Handbook of International Economics,* Vol. I, Amsterdam, North-Holland.
JONES, R. W. and SCHEINKMAN, J. (1977). The relevance of the two-sector production model in trade theory, *Journal of Political Economy,* October.
MCFADDEN, D. (1978). Cost, revenue and profit functions, in FUSS, M. and MCFADDEN, D. (eds), *Production Economics: A Dual Approach to Theory and Applications,* Amsterdam, North-Holland.
METCALFE, J. S. and STEEDMAN, I. (1972). Reswitching and primary input use, *Economic Journal,* March.
PASINETTI, L. (1977). *Lectures on the Theory of Production,* New York, Columbia University Press.
ROY, R. (1942). *De l'utilité,* Paris, Hermann.
RYBCZYNSKI, T. M. (1955). Factor endowments and relative commodity prices, *Economica,* November.
SAMUELSON, P. A. (1953–54). Prices of factors and goods in general equilibrium, *Review of Economic Studies,* October (No. 54).
SHEPHARD, R. W. (1953). *Cost and Production Functions,* Princeton, Princeton University Press.
SHEPHARD, R. W. (1970). *Theory of Cost and Production Functions,* Princeton, Princeton University Press.
STOLPER, W. F. and SAMUELSON, P. A. (1941). Protection and real wages, *Review of Economic Studies,* November.

APPENDIX A

Shephard's lemma, applied to a constant returns to scale technology, yields the following derivatives of the $m+n$ cost functions in our model:

$$
\begin{aligned}
&\partial[\mathbf{w}'\beta+(1+r)\mathbf{p}'a]/\partial\mathbf{w}'=\beta,\\
&\partial[\mathbf{w}'\beta+(1+r)\mathbf{p}'a]/\partial\mathbf{p}'=(1+r)a,\\
&\partial[\mathbf{w}'b+(1+r)\mathbf{p}'a]\partial w'=b,\\
&\partial[\mathbf{w}'b+(1+r)\mathbf{p}'a]/\partial\mathbf{p}'=(1+r)a,
\end{aligned}
\tag{A.1}
$$

taking r as given. In each equation a row vector of cost functions (m such functions in the first two equations and n in the last two) is differentiated by a row vector of prices (q factor prices in the first and third equations and n capital good prices in the second and fourth equations). The derivative of the jth cost function with respect to the ith price is the element in the ith row and jth column of the matrix on the right-hand size.

It follows from (A.1) that, in the total differential of the price equations (3) and (4),

$$d\pi' = (d\mathbf{w}')\beta + (d\mathbf{p}')(1+r)a + \mathbf{w}'(d\beta) + \mathbf{p}'(1+r)(da), \tag{A.2}$$
$$d\mathbf{p}' = (d\mathbf{w}')b + (d\mathbf{p}')(1+r)a + \mathbf{w}'(db) + \mathbf{p}'(1+r)(da), \tag{A.3}$$

the last two terms in each equation vanish.[1] It is an implication of cost minimisation that these sums are zero, a result that Samuelson dubbed the Wong–Viner envelope theorem. Thus, Shephard's lemma implies that in the neighbourhood of a cost-minimising point, the effects of changes in **w** and **p** on the technique of production can be ignored: 'The substitution effects are of a higher order of smallness, influencing curvatures rather than slopes' (Samuelson, 1953–54, p. 5). Eliminating $d\mathbf{p}'$ in the above equations,

$$d\pi' = (d\mathbf{w}'\beta + (d\mathbf{w}')b[I-(1+r)a]^{-1}(1+r)a = (d\mathbf{w}')C, \tag{A.4}$$

so that, in the total differential of (12),

$$d\pi' = (d\mathbf{w}')C + \mathbf{w}'(dC), \tag{A.5}$$

it is the second price-weighted sum of coefficient changes which is zero at a cost-minimising point. This might appear to be nothing more than the condition that an isoquant (in q dimensions) defining direct and indirect factor requirements must be tangent to a cost plane at a point of minimum factor cost. In fact, it is not this condition (unless $r=0$) since the quantities of direct and indirect factors per unit of output of each commodity are given by the elements of B rather than C.

Shephard's lemma does not permit the differential form of the quantity equations to be simplified in the same way as in the price equations. First, differentiate (11) to obtain

$$(dB)\mathbf{x} + B(d\mathbf{x}) = d\mathbf{v}. \tag{A.6}$$

A typical first sum on the left side of equation i in (A.6) may be written

$$\sum_j (dB_{ij})x_j = \sum_j x_j \sum_k (\partial B_{ij}/\partial w_k)dw_k = \sum_k s_{ik}dw_k, \tag{A.7}$$

where

$$s_{ik} = \sum_j x_j(\partial B_{ij}/\partial w_k) \tag{A.8}$$

is the element in row i and column k of a square substitution matrix S of dimension q. Note that the coefficients in the quantity equations, B_{ij}, vary whenever the coefficients in the price equations, C_{ij}, vary. The latter are functions of **w** and r, once **p** has been eliminated from the price equations. The profit rate is given and so each B_{ij} is a function of **w** alone. But, since the weights on the derivatives summed in (A.8) are commodity outputs instead of factor prices, the s_{ik} do not vanish (irrespective of the value of r). We are left, after substituting (A.7) into (A.6), with

$$S(d\mathbf{w}) + B(d\mathbf{x}) = d\mathbf{v} \tag{A.9}$$

[1] Readers of Hicks (1965), whose notation we are following, will recall that da and db are not defined in his analysis where a change of technique *means* a change in the physical characteristics of produced inputs (and hence their units of measurement) rather than simply a change in the proportions in which various produced and non-produced inputs are used. But here we follow neoclassical convention.

which, together with (A.4), may be written[1]

$$\begin{pmatrix} S & B \\ C' & 0 \end{pmatrix} \begin{pmatrix} d\mathbf{w} \\ d\mathbf{x} \end{pmatrix} = \begin{pmatrix} d\mathbf{v} \\ d\boldsymbol{\pi} \end{pmatrix}. \tag{A.10}$$

The usual interpretation of a substitution term like s_{ik} is to say that when the price of the service of factor k goes up there is a substitution away from the use of that factor towards the use of other factors. Consequently, in neoclassical models, s_{kk} are negative and at least some s_{ik} are positive, $i \neq k$; $i, k = 1, \ldots, q$. Complementarities (negative s_{ik}, $i \neq k$) may exist but are generally assumed to be dominated by substitution effects. The point we wish to make about the consequences of a positive rate of profit is a different one. Thus, suppose we assume a regular minimum for the cost function in every industry (given a positive r). Therefore $\sum_i \sum_k z_i z_k \partial C_{ij}/\partial w_k$ is negative for each j and every vector $\mathbf{z}$ not proportional to $\mathbf{w}$. But *only* if $r = 0$ would it follow that S has a negative diagonal and rank equal to $q-1$ (Jones, 1979, p. 114). For, in that event, $B = C$ and so changes in the coefficients of C associated with cost-minimising reactions to changes in $\mathbf{w}$ are exactly the same as the changes in the corresponding elements of B. But, when r is positive, this is not so. A rise in the price of a given factor service may not be associated with a reduced demand for it in terms of the factor requirements appropriate to the quantity equations. This generalises the argument in Metcalfe and Steedman (1972) to the case of more than two factors.

APPENDIX B

A simple linear programming model can be used to illustrate Samuelson's 'reciprocity relation'.

PRIMAL PROBLEM: Maximise $\pi_1 x_1 + \pi_2 x_2 = 36x_1 + 30x_2$
subject to $\beta_{11}x_1 + \beta_{12}x_2 = 6x_1 + 2x_2 \leqslant 120 = v_1$,
$\beta_{21}x_1 + \beta_{22}x_2 = 3x_1 + 4x_2 \leqslant 96 = v_2$.

The solution for outputs is $(x_1, x_2) = (16, 12)$ with both resource constraints satisfied as equations. Changing only v_1 to 156 makes the solution $(x_1, x_2) = (24, 6)$; changing only v_2 to 114 makes the solution $(x_1, x_2) = (14, 18)$. Thus, at constant commodity prices, we have four partial effects:

$$\Delta x_1/\Delta v_1 = 2/9; \quad \Delta x_2/\Delta v_1 = -1/6; \quad \Delta x_1/\Delta v_2 = -1/9; \quad \Delta x_2/\Delta v_2 = 1/3. \tag{B.1}$$

DUAL PROBLEM: Minimise $w_1 v_1 + w_2 v_2 = 120w_1 + 96w_2$
subject to $w_1\beta_{11} + w_2\beta_{21} = 6w_1 + 3w_2 \geqslant 36 = \pi_1$,
$w_1\beta_{12} + w_2\beta_{22} = 2w_1 + 4w_2 \geqslant 30 = \pi_2$.

[1]Compare with Jones and Scheinkman (1977, p. 926), where S has neoclassical properties and C' is the transpose of B.

The solution for prices is $(w_1, w_2)=(3, 6)$ with both price constraints satisfied as equations. Changing only π_1 to 63 makes the solution $(w_1, w_2)=(9, 3)$; changing only π_2 to 36 makes the solution $(w_1, w_2)=(2, 8)$. Thus, given factor supplies, we have four more partial effects:

$$\Delta w_1/\Delta\pi_1=2/9; \qquad \Delta w_2/\Delta\pi_1=-1/9; \qquad \Delta w_1/\Delta\pi_2=-1/6; \qquad \Delta w_2/\Delta\pi_2=1/3. \quad \text{(B.2)}$$

Comparing (B.1) and (B.2) confirms Samuelson's 'reciprocity relation'. In this example, commodity 1 is relatively intensive in its use of factor 1. The factor supply shifts and commodity price changes affect commodity outputs and factor prices in the 'magnified' way referred to in the text.

In addition to primary factor inputs, let there now be two produced inputs and, hence, four sectors in all. The technique matrix is

$$\begin{pmatrix} \beta_{11} & \beta_{12} & b_{11} & b_{12} \\ \beta_{21} & \beta_{22} & b_{21} & b_{22} \\ a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 6 & 2 & 0{\cdot}2 & 1 \\ 3 & 4 & 1 & 0{\cdot}2 \\ 1{\cdot}5 & 0 & 0{\cdot}5 & 0 \\ 0 & 1 & 0 & 0{\cdot}5 \end{pmatrix}$$

The zeros indicate that capital goods are specific to final commodity sectors. This merely simplifies the calculations. Using the definition of B in (11),

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 6{\cdot}6 & 5 \\ 5 & 4{\cdot}4 \end{pmatrix}$$

In the quantity equations, commodity 1 uses factor 1 (directly and indirectly) relatively intensively since $B_{11}/B_{21}-B_{12}/B_{22}=4{\cdot}04>0$. Using the definition of C in (12), on the other hand, with $r=10\%$, yields

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 6{\cdot}73 & 4{\cdot}44 \\ 6{\cdot}66 & 4{\cdot}48 \end{pmatrix},$$

where all final decimals repeat. Thus, in the price equations, the cost of factor 1 (inclusive of indirect costs compounded at the rate of 10%) is relatively greater in sector 1, since $C_{11}/C_{21}-C_{12}/C_{22}=0{\cdot}58$, but the difference between these two ratios is less than the difference between the corresponding coefficients in the quantity equations. Using the B_{ij} in the price equations would understate the effect of commodity price changes on factor prices; using the C_{ij} in the quantity equations would overstate the effect of factor supply changes on the composition of output. The problem is seen even more clearly when $r=20\%$. In that event,

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 6{\cdot}9 & 6 \\ 6{\cdot}5 & 4{\cdot}6 \end{pmatrix}.$$

Now $C_{11}/C_{21}-C_{12}/C_{22}=-7{\cdot}26<0$, so that the cost of factor 1 (compounded at 20%) is relatively greater in sector 2. The reciprocal effects of price changes on income distribution and of factor supply changes on the composition of output are not only unequal but of opposite sign.

Allowing for a choice of production processes in each sector raises a great many possibilities. Two examples follow. The first involves a minimal change in our example. Let $\beta_{11}=6{\cdot}1$ and $\beta_{21}=1{\cdot}9$ in the last example. We therefore have an additional technique matrix. It can then be verified that if the factor supplies are $(v_1, v_2)=(125, 100)$, the value-maximising composition of output is $(x_1, x_2)=(12{\cdot}4, 8{\cdot}6)$ when the price ratio is $\pi_1/\pi_2=1{\cdot}22$, and requires that the original technique be used. For a higher price ration $\pi_2/\pi_2=1{\cdot}3$, the value-maximising outputs are $(x_1, x_2)=(10, 11{\cdot}5)$ and it is the new technique which must be used. Meanwhile, factor prices have changed from a ratio of $w_1/w_2=2$ to a ratio of $0{\cdot}5$. Factor substitution is conventional: the new process in the first sector uses proportionally more of factor 1 and, when it is used, the corresponding relative factor price is lower. However, the output response is not conventional: when π_1/π_2 is higher, the output ratio x_1/x_2 is lower. The reason is that, in minimising factor cost at the new product price ratio, the system adopts a technique that saves on the second factor since it is this factor which contributes relatively heavily to the cost of the commodity which has increased in value, commodity 1. But, as far as production is concerned, it is commodity 2 which uses factor 2 relatively intensively and so it is that sector which must expand in order to maintain equality in the resource constraints.

Our final example, based on Steedman and Metcalfe (1972), is important for showing non-conventional substitution possibilities. The 4×4 matrix written below is similar to the one in the previous example except that now each capital good, though required in only one final good sector, requires the other capital good (and possibly itself) in its own production. The following matrix replaces the previous one with the numbers in parentheses indicating an alternative process for producing the first capital good. Once again, we have two techniques with a single process differing between them.

$$\left(\begin{array}{c|c} \beta & b \\ \hline a & a \end{array}\right) = \left(\begin{array}{cc|cc} 600 & 200 & 1\,(66) & 100 \\ 300 & 400 & 100\,(120) & 100 \\ \hline 1 & 0 & 0\,(0{\cdot}51) & 0{\cdot}17 \\ 0 & 1 & 1{\cdot}12\,(0{\cdot}03) & 0 \end{array}\right)$$

As before, we write out B and C, assuming $r=20\%$. For the first technique,

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 740 & 324 \\ 562 & 545 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 824 & 366 \\ 688 & 599 \end{pmatrix},$$

and for the second technique

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 742 & 324 \\ 553 & 543 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 819 & 365 \\ 690 & 600 \end{pmatrix}$$

Given factor supplies in the ratio $v_1/v_2=2/3$, it can then be shown that at the price ratio $\pi_1/\pi_2=1{\cdot}3$ the first technique is cost minimising, the factor price ratio is $w_1/w_2=0{\cdot}26$, and the composition of output is in the ratio $x_1/x_2=0{\cdot}108$. At a higher price ratio, $\pi_1/\pi_2=1{\cdot}5$, the second technique is cost minimising, the factor price ratio has increased to 0·77, and the output ratio has decreased to 0·102. This, again, indicates an unconventional supply response: a shift away from x_1 towards x_2 when the relative price of x_1 is higher. This is explained by the remarkable fact that in this example, comparing the two techniques, all the elements of B move in the opposite direction to the corresponding elements of C (the exception appearing constant because of rounding). At the higher commodity price ratio, the second technique is substituted in place of the first, and yet, because this increases the requirements of factor 1 relative to factor 2 in production, it is necessary (in order to maintain equality in the resource constraints) to shift production towards commodity 2, which (in both techniques) uses relatively less of the first factor compared to commodity 1.

[13]

RENT THEORY IN A MULTISECTORAL MODEL[1]

By HEINZ D. KURZ

I. Introduction

PIERO SRAFFA's *Production of Commodities by Means of Commodities*[2] has so far been almost exclusively interpreted as a contribution to the critique of the marginal theory of value and distribution. This interpretation is obvious because criticism concentrates upon the notion of 'capital', hypothesized by neoclassical writers conceiving of capital simply as a *quantity* that can be measured independently of, and prior to, the determination of the prices of the means of production. However, 'capital' being a central concept in economics, a critique of the prevailing neoclassical capital theory simultaneously is a critique of economic theory as a whole (at least a prelude to it, as Sraffa indicates in the sub-title of his book). This paper discusses Sraffa's critique of *rent* theory in the short eleventh chapter, 'Land'.[3] A comparison with classical rent theory as formulated, in particular, by Ricardo brings out a close affinity of the two approaches such that Sraffa's analysis may be called 'neo-Ricardian'.

The method adopted in this paper is that of comparative static analysis of stationary economies which was masterly handled by Adam Smith, Ricardo, and Marx.[4] As is known, Sraffa was interested in dealing only with those properties of an economic system that do not depend on changes in the scale of production or on the proportions of 'factors'. The prices to be found in his theory therefore have the character of *long-run equilibrium prices* and thus are closely related to the concept of 'natural price', of which Ricardo writes: 'By the very definition of natural price, it is wholly dependent on cost of production, and has *nothing* to do with demand and supply.'[5] At first sight this concept seems to hinder the possibility of adequately

[1] I am most grateful to A. Jeck, H. Hagemann, R. Tilley, S. Hübner, and an anonymous referee of the *O.E.P.* for helpful suggestions. Any errors are, of course, my own.

[2] Cambridge, 1960.

[3] As far as I know, only the following authors have so far applied Sraffa's approach to the problem of the rent of land: A. Quadrio Curzio, *Rendita e distribuzione in un modello economico plurisettoriale*, Milan, 1967; G. Montani, 'La teoria ricardiana della rendita', *L'industria*, 1972, pp. 221–43; J. S. Metcalfe and Ian Steedman, 'Reswitching and primary input use', *Economic Journal*, vol. 82, 1972, pp. 140–57.

[4] Cf. P. Garegnani, 'Heterogeneous capital, the production function and the theory of distribution', *Review of Economic Studies*, vol. 37, 1970, p. 427.

[5] Ricardo in his letter to Trower of 21 July 1820, *The Works and Correspondence of David Ricardo*, edited by P. Sraffa with the collaboration of M. H. Dobb, Cambridge, 1951–73 (in this paper quoted as Ricardo I, II, etc.), Ricardo VIII, p. 207. Italics, if not otherwise stated, are my own.

investigating natural resources, particularly land, within the framework of value and distribution theory. The analysis of rent depends crucially upon the requirement that the output produced by means of land can vary; so *demand* necessarily must be taken into account. Malthus already objected to Ricardo:

> When you reject the consideration of demand and supply in the price of commodities . . . you appear to me to look only at half of your subject. . . . How is the price of corn, and the quality of the last land taken into cultivation determined but by the state of the population and the demand ?[1]

In order to evade the demand problem we may assume the quantities produced of the various products to be given and to be compatible with a state of simple reproduction. Provided that the wage rate (rate of profits) is exogenously fixed, for a particular given output vector, we can find the system of relative prices, the rate of profits (wage rate), and the rent rates associated with this level of production. Thereupon we may hypothetically vary the produced quantities, particularly those of agricultural products, and again determine the values of the above variables to compare them with the values obtained in the first situation. The methodological continuity of Sraffa's approach manifests itself in the very fact that his analysis of land essentially does not differ from his analysis of the pure capital model. Sraffa stresses that the assumption of constant returns to scale should not be attributed to his model without unproduced means of production. It is therefore obvious that all results obtained are valid only at a *particular given level of production.*[2]

Part II is dedicated to a presentation of the neo-Ricardian theory of extensive rent. In its first section the price system of an economy cultivating land of k different qualities will be discussed; in the second section we try to find out whether or not the ranking of those k qualities of land according to their fertility corresponds to their ranking according to the rents they yield. Part III presents the theory of intensive rent. Its first section describes the price system in the case of land of homogeneous quality cultivated at different degrees of intensity; in the second section we deal with variations in the rent per acre and in the rate of profits as a consequence of the intensification of cultivation. Part IV summarizes briefly the argument and gives further examples where we can apply the neo-Ricardian rent theory.

[1] Malthus in his letter to Ricardo of 26 Oct. 1820, Ricardo VIII, p. 286. Cf. also Ricardo's anti-critique, Ricardo I, pp. 405–6 and 409. Indeed, in *short-term* analysis Ricardo does consider the role demand plays in the determination of (market) prices and sketches demand functions, e.g. for corn; see, for example, Ricardo IV, p. 220.

[2] As J. Robinson puts it, 'we are given only half of an equilibrium system to stand on' ('Prelude to a critique of economic theory', vol. 13 of this journal, 1961, p. 54).

II. Differential rent as a form of extensive rent

1. The price system with k different qualities of cultivated land

1. Let us suppose in what follows that the economy is divided into two main sectors, the industrial and the agricultural sector. For simplicity, land is assumed to be used only in agriculture. The industrial sector consists of n single-product industries each one of them producing one specific commodity. In agriculture, one homogeneous product g, called *corn*, is grown. All the $n+1$ products are *basic* products, that is they enter directly or indirectly into the production of *all* commodities. Corn is grown on k different qualities of land out of s ($s > k$); there is a specific method of production associated with each quality of land.[1] The complete system therefore consists of $n+k$ processes. Let a_{ij} $(i,j = 1, 2,..., n)$ designate the amount of the industrial product i, a_{gj} the amount of corn, and l_j the amount of direct labour that enter the production of b_j units of the industrial product j; let a_{ig}^z $(z = 1, 2,..., k)$, a_{gg}^z and l_g^z designate the respective amounts of inputs in the production of b_g^z units of corn produced on land Λ^z, and ρ^z the corresponding rent per acre; p_j is the price of one unit of commodity j, p_g the price of corn, w the uniform wage rate, and r the uniform rate of profits. The price equations, therefore, may be written as follows:[2]

$$
\left.
\begin{array}{l}
\left.
\begin{array}{c}
(1+r)(a_{11}p_1+...+a_{n1}p_n+a_{g1}p_g)+wl_1 = b_1p_1 \\
(1+r)(a_{12}p_1+...+a_{n2}p_n+a_{g2}p_g)+wl_2 = b_2p_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(1+r)(a_{1n}p_1+...+a_{nn}p_n+a_{gn}p_g)+wl_n = b_np_n
\end{array}
\right\} \text{(I)} \\
\left.
\begin{array}{c}
(1+r)(a_{1g}^1p_1+...+a_{ng}^1p_n+a_{gg}^1p_g)+wl_g^1+\rho^1\Lambda^1 = b_g^1p_g \\
(1+r)(a_{1g}^2p_1+...+a_{ng}^2p_n+a_{gg}^2p_g)+wl_g^2+\rho^2\Lambda^2 = b_g^2p_g \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(1+r)(a_{1g}^kp_1+...+a_{ng}^kp_n+a_{gg}^kp_g)+wl_g^k+\rho^k\Lambda^k = b_g^kp_g.
\end{array}
\right\} \text{(II)}
\end{array}
\right\} (1)
$$

There are $n+k+3$ unknowns (n prices of industrial products, the price of corn, k rents, w, and r) contained in $n+k$ equations. If we standardize the system by setting the price of corn p_g equal to unity, i.e. by expressing all the values in quantities of corn, and if we—following Ricardo's lead—assume the wage rate w to be exogenously given, there remains one degree of freedom. To eliminate it we must assume that one quality of land, *marginal land*, does not yield a rent. 'By this token only can it [i.e. marginal

[1] The way they are connected could be strictly technological, or, more realistically, it can be the result of an economic selective process carried out for a given wage rate (rate of profits).

[2] The fact that cultivable land itself generally is a *produced* means of production raises the problem of dividing rent between profit on *la terre-capital*, as Marx used to say, and rent in its proper sense. For brevity, we shall not deal with this problem here.

land] be identified as the least productive land in use.'[1] This assumption provides us with another independent equation,

$$\prod_{z=1}^{k} \rho^z = 0, \tag{2}$$

whereupon system (1) can be solved. Let the least productive land, which is supposed to be available in excess, be the land k, i.e. $\rho^k = 0$. The corresponding price equation then assumes a pattern that coincides with the pattern of the industrial price equations:

$$(1+r)(a_{1g}^k p_1 + \ldots + a_{ng}^k p_n + a_{gg}^k p_g) + w l_g^k = b_g^k p_g. \tag{3}$$

(3) is the price equation of corn. Together with the n equations of the industrial products (1/I), it determines the system of the $n+1$ prices—*independent* of the other agricultural processes. (3) in connection with (1/I) may therefore be called the *price determination system*. When we put the prices resulting from it into the first $k-1$ equations of the agricultural sector, we obtain the rents $\rho^1, \rho^2, \ldots, \rho^{k-1}$.

2. Equation (3) reflects an old Ricardian finding, which, in terms of the labour theory of value,[2] says:

> The value of corn is regulated by the quantity of labour bestowed on its production on that quality of land, or with that portion of capital, *which pays no rent. Corn is not high because a rent is paid, but a rent is paid because corn is high*; . . . no reduction would take place in the price of corn, although landlords should forego the whole of their rent. Such a measure . . . would not diminish the quantity of labour necessary to raise raw produce on the least productive land in cultivation.[3]

Elsewhere Ricardo says:

> If the high price of corn were the *effect*, and not the *cause* of rent, price would be proportionally influenced as rents were high or low, and rent would be a component part of price. But that corn which is produced by the greatest quantity of labour is the regulator of the price of corn; and rent does not and cannot enter in the least degree as a component part of its price.[4]

Ricardo's last remark is evidently aimed at Adam Smith's 'adding-up theory' of relative prices, according to which all prices are composed of wage, profit, *and* rent and react to autonomous changes in each one of these components by varying in the *same* direction.[5] Instead, Ricardo advocates

[1] Sraffa, p. 74 n. 1.

[2] i.e., strictly speaking, provided that $r = 0$; see Sraffa, p. 12.

[3] Ricardo I, pp. 74–5.

[4] Ricardo I, p. 77; similarly ibid., pp. 72, 78, 212–13, 284, 399–400, 409, and Ricardo IV, pp. 210–12.

[5] Cf. A. Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations*, two volumes, edited and introduced by E. Cannan, fourth edition, London, 1925, vol. i, pp. 49–54, particularly p. 51. For the sake of completeness it should be mentioned that Smith in a passage of the 'Wealth of Nations' comes very close to the Ricardian point of view: 'Rent . . . enters into the composition of the price of commodities in a different way from wages and profit. High or low wages and profit are the causes of high or low price; *high or low rent is the effect of it*' (ibid., p. 147).

the thesis about differential rent in its extensive form that is substantiated by (3), saying that rent does not take part in price determination, and concludes:

> Adam Smith, therefore, cannot be correct in supposing that the original rule which regulated the exchangeable value of commodities, namely, the comparative quantity of labour by which they were produced, can be at all altered by the appropriation of land and the payment of rent.[1]

Hence the price of corn *finally* resolves itself only into two parts, wages and profits.[2]

3. Following Ricardo and Sraffa, we have so far assumed that there is land of s different qualities, k of which are in cultivation. Moreover, we have assumed that among those k qualities the kth is the lowest, and will, therefore, be agriculturally utilized last. But how do we know which land is of the highest and which is of the lowest quality? Generally speaking: How can we rank land of varying quality according to its fertility, so as to indicate the order in which these varieties of land are taken into cultivation to satisfy a growing social demand for corn?

In a capitalist society this question is settled—for a given wage rate—by following the criterion of maximizing the rate of profit. In order to determine the maximum rate of profit, the second-best, and so on, we successively treat each one of the various qualities of land as if it were the *marginal* one, i.e. we see to it that its corresponding process equation conforms in its structure with (3). This way we obtain as many price determination systems of the type (1/I)+(3) as there are qualities of land, each one of them consisting of $n+1$ equations, and just as many rates of profit.[3] The ordering of these rates of profit corresponds to the 'fertility ordering' that we are trying to find. The tenant-capitalist goes by this ordering when taking into cultivation more and more land to satisfy a growing demand for agricultural products. He begins cultivating the highest-quality land, which yields the highest rate of profit, then goes on to the second-best land, and so on, until even the lowest-quality land is under cultivation. If it is not possible to intensify the cultivation by increasing expenditure on capital and labour to yield increasing returns per acre (cf. Part III of this

[1] Ricardo I, pp. 77–8.

[2] See, for example, Ricardo I, p. 409. Thus, Ricardo correctly criticizes Smith concerning extensive rent; but in this case there is no justification for Marx criticizing Ricardo and, in contrast to (3), asserting that for the determination of prices those commodities are decisive 'which are produced under the *medium*, the *average conditions* of this sphere. By no means under the *worst conditions*, as Ric[ardo] supposes in connexion with rent' (K. Marx, *Theorien über den Mehrwert*, three volumes, edition: Marx–Engels Werke, vol. 26. 1–3, Berlin, 1971; here vol. 26. 2, p. 191, Marx's italics; cf. also pp. 203–4. All translations from German are mine.)

[3] We assume that at the given wage rate w all the qualities of land yield *positive* rates of profit.

paper), and if there is no technical progress in agriculture,[1] the production of corn can no longer be expanded when the whole area of the worst-quality land is under cultivation.

4. Ricardo obviously presumes that this hierarchy of land according to fertility is *naturally* given and—without investment in amelioration—unalterable.[2] Marx is of the same opinion when saying that the fertility of land as well as its location is 'independent of capital', and that it represents an 'objective property of land'.[3] *Sraffa on the contrary stresses the fact that the ranking according to productivity depends on the actual w–r combination.*[4] In other words: The ordering of land of different qualities determined at a wage rate w^* need not coincide with that ordering associated with wage rate w^{**} ($w^* \neq w^{**}$). Consequently, at a given demand for corn it cannot be known in advance which qualities of land will be under cultivation and, therefore, which will be scarce. The reason for this surprising result *contradicting the classical as well as Marxian rent theory* is that the prices of the $n+1$ products are in general complicated functions of the wage rate.[5] When there are (hypothetical) changes in the wage rate the costs of growing corn on the various qualities of land will change (price Wicksell effects), so that a certain quality, say Λ^l, which is more profitable at a wage rate w^*, may at the wage rate w^{**} give way to the formerly less profitable quality Λ^m. This *phenomenon of changes in the ordering according to profitability* can be elucidated through the following argument. As is well known, each one of the (1/I)+(3)-systems has its own w–r relationship. So there are as many distribution curves as there are qualities of land. When we sketch the w–r relationships associated with the qualities Λ^l and Λ^m in a diagram (see Fig. 1) the following combinations are conceivable: Fig. 1a represents the classical and Marxian point of view. Over the whole range of variation of the wage rate, cultivating Λ^l is more profitable than cultivating Λ^m. In Fig. 1b, it is more profitable to grow corn on quality Λ^l when the wage is w^*, whereas at the wage w^{**} quality Λ^m is more profitable. A transition from w^* to w^{**} entails a change in the method of production and in the quality of land. In Fig. 1c, at both wages, w^* as well as w^{**}, Λ^l is preferred; in a certain interval between w^* and w^{**}, however, Λ^m turns out to be more advantageous. During a gradual transition from w^* to w^{**}, we observe the

[1] In this paper, technical progress is excluded by assumption. A detailed analysis of the impact of technical progress on rents, the share of rents, the rate of profits, and the prices of agricultural products is found in Ricardo's work; cf. particularly Ricardo I, pp. 79–84, and Ricardo IV, p. 19 and p. 32.

[2] See, for example, Ricardo I, pp. 70–1.

[3] K. Marx, *Das Kapital*, three volumes, edition: Marx–Engels Werke, vols. 23–5, Berlin, 1969, here *Das Kapital* III, vol. 25, pp. 663–4.

[4] Sraffa, p. 75.

[5] See, for example, Sraffa, particularly chap. VI, and B. Schefold, 'Relative prices as a function of the rate of profit: a mathematical note', *Zeitschrift für Nationalökonomie*, vol. 36, 1976, pp. 21–48.

method of production associated with Λ^m 'temporarily' dominate but lastly the method associated with Λ^l return.[1]

5. We have just seen how the various qualities of land can be ranked according to fertility at a *given* wage rate. Now we shall investigate the variation in the prices of the $n+1$ products as well as in the rate of profits caused by the (purely hypothetical) 'transition' from the best to worse and worse qualities of land because of growing demand for corn. For this purpose, let us assume that the qualities of land Λ^z $(z = 1, 2, \ldots, k)$, the most fertile land being Λ^1, have already been ranked as described above.

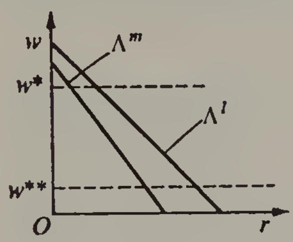


FIG. 1a.

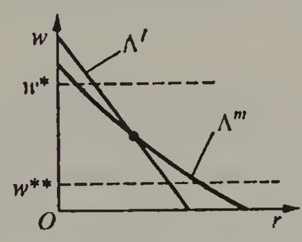


FIG. 1b.

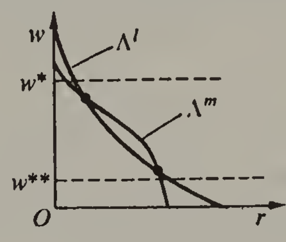


FIG. 1c.

As long as it is sufficient to satisfy the given demand for corn by cultivating the available area of the highest-quality land Λ^1, and as long as this land is not made scarce 'artificially', there can be no rent.[2] In this case the agricultural sector consists of a single process, whose price equation conforms to (3), and which together with the n industrial processes determines the system of relative prices and the rate of profits. When the demand for corn exceeds the amount that can be grown on the whole available area of the first-quality land, second-quality land, which yields a lower rate of profits, must be taken into cultivation. Simultaneously, Λ^1 passes its position as *marginal* land on to Λ^2. By this time the agricultural sector consists of two processes:

$$(1+r)(a^2_{1g}p_1+\ldots+a^2_{ng}p_n+a^2_{gg}p_g)+wl^2_g = b^2_g p_g$$
$$(1+r)(a^1_{1g}p_1+\ldots+a^1_{ng}p_n+a^1_{gg}p_g)+wl^1_g+\rho^1\Lambda^1 = b^1_g p_g.$$

The second equation contains the statement that from now on the first-quality land will yield a rent. In Ricardo's words:

> When in the progress of society, land of the second degree of fertility is taken into cultivation, rent immediately commences on that of the first quality, and the amount of that rent will depend on the difference in the quality of those two portions of land.[3]

[1] In the literature on *capital theory* this last case is known as 'reswitching of technique'; cf., for example, Sraffa, part III, and Garegnani, pp. 410–14. The maximum number of possible switches between the two production systems is equal to the number of basic products common to both systems, i.e. $n+1$; see K. Bharadwaj, 'On the maximum number of switches between two production systems', *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, vol. 106, 1970, pp. 409–29.

[2] See, for example, Ricardo I, p. 69.

[3] Ricardo I, p. 70.

The cause of the existence of rent is the fact that in the new situation the vector of the industrial prices $p = [p_1, p_2, ..., p_n]$ expressed in terms of corn quantities ($p_g \equiv 1$) is strictly *smaller* than the price vector of the first situation when only Λ^1 was cultivated. That is to say that at a constant real wage rate w expressed in terms of corn quantities and a lower rate of profits r the equation associated with Λ^1 can include a positive rent per acre ρ^1, because the industrial prices have fallen relatively to the price of corn. In the context of the labour theory of value this argument runs as follows:

> The reason then, why raw produce rises in comparative value, is because *more labour* is employed in the production of the last portion obtained, and not because a rent is paid to the landlord.[1]

If the demand for corn continues increasing, the society eventually must resort to even less fertile land Λ^3. So the industrial priccs expressed in terms of corn quantities fall again. Thereupon second-quality land also becomes intramarginal and yields a rent, whereas the rent on first-quality land rises:

> With every step in the progress of population, which shall oblige a country to have recourse to land of a worse quality, to enable it to raise its supply of food, rent, on all the more fertile land, will rise.[2]

2. *Ranking according to profitability and according to rent per acre*

6. The view advocated by Ricardo and Marx that the ranking of the various qualities of land according to their fertility (i.e. profitability) is *naturally* given and unalterable without investment in amelioration, has been shown to be not generally valid. *The ordering in question depends on distribution.* In what follows we shall investigate the additional Ricardian/Marxian thesis that the ranking of land according to fertility corresponds to the ranking according to the associated rents per acre, such that the 'most fertile' land always yields the highest rent, the second-best land the second highest, and so on. Ricardo does not leave any room for doubt that this thesis is correct:

> When land of the third quality is taken into cultivation, rent immediately commences on the second, and it is regulated as before, by the difference in their productive powers. At the same time, the rent of the first quality will rise, *for that must always be above the rent of the second*, by the difference between the produce which they yield with a *given quantity of capital and labour*.[3]

Marx shows this supposed congruity by means of several numerical examples, and points out that 'the ratio of the rents yielded by the various

[1] Ricardo I, p. 74; cf. also ibid., pp. 73, 75–6, 93–4, and Ricardo IV, p. 212.
[2] Ricardo I, p. 70; cf. also K. Marx, *Das Kapital* III, vol. 25, pp. 665–80.
[3] Ricardo I, p. 70.

qualities of land (reckoned per acre), and therefore also the rent rate in relation to the capital invested per acre, remain unchanged'.[1]

7. The Ricardian and Marxian point of view is correct as long as it is assumed that on each one of the k different qualities of land the *same* aggregate means of production and the *same* amount of labour per acre are employed. Ricardo seems to have this technical premiss in mind when talking about '*a given quantity of capital and labour*' as quoted above. In terms of the neo-Ricardian system, this means that the k process-specific input vectors contained in (1/II), expressing the means of production and labour requirements, i.e.

$$v^z = [a^z_{1g}, a^z_{2g}, \dots, a^z_{ng}; a^z_{gg}; l^z_g] \quad (z = 1, 2, \dots, k),$$

are pairwise linearly dependent such that

$$\frac{1}{\Lambda^1} v^1 = \frac{1}{\Lambda^2} v^2 = \dots = \frac{1}{\Lambda^k} v^k.$$

In this case it is excluded *ex hypothesi* that, in the course of expansion of the corn production and the changes in distribution and prices associated with it, the capital values of the aggregated means of production employed on the various qualities of land vary in *opposite* directions: the variations in prices have the same effects on each of the k agricultural processes. Thus the preliminary condition discussed in the following paragraph, which provides that the orderings according to profitability and according to rents may diverge, is omitted.

8. If we assume, however—in contrast to Ricardo and Marx and in accordance with Sraffa—that *different* qualities of land generally require *different* agricultural methods of production (characterized by the input of different means of production, or of the same means of production in different proportions, and of different amounts of direct labour), Ricardo's and Marx's point of view is no longer valid. As Ricardo himself indicates in another context, the reason is that the *value* of the means of production is not independent of distribution:

> I would ask what means you have of ascertaining the equal value of capitals? . . . *These capitals are not the same in kind*—what will employ one set of workmen, is not precisely the same as will employ another set, and if they themselves are produced in unequal times *they are subject to the same fluctuations as other commodities*.[2]

As soon as the aggregate means of production employed on the various qualities of land are not physically identical, it is neither guaranteed that their respective total values are equivalent at a given wage rate (rate of profits), nor that they vary in the same direction, when distribution changes.

[1] K. Marx, *Das Kapital* III, vol. 25, p. 677; cf. also p. 671.

[2] Ricardo in his letter to McCulloch of 21 Aug. 1823, Ricardo IX, pp. 359–60; cf. also Ricardo IV, pp. 393–4.

Let us take a closer look at this case.[1] We know that as a result of gradually extending corn production to worse and worse land the rents yielded by land of better quality will rise. Initially let Λ^3 be the marginal land. The rent per acre yielded by Λ^1 in this situation amounts to $\rho^1_{(3)}$ and should exceed the rent $\rho^2_{(3)}$ at the same time yielded by Λ^2. If Λ^4 is taken into cultivation to satisfy a growing demand for corn, the prices of the industrial products will fall relatively to the price of corn, whereas the rents of the best and second-best qualities will rise to $\rho^1_{(4)}$ and $\rho^2_{(4)}$ respectively. According to Ricardo and Marx, $\rho^1_{(4)}$ should be higher than $\rho^2_{(4)}$ just as before $\rho^1_{(3)}$ exceeded $\rho^2_{(3)}$. This conclusion, however, does not hold generally. Whether or not the order of the rents is still retained after such changes crucially depends on the changes in the prices of the various means of production, and on the quantities and proportions in which they are employed on the various qualities of land. It is conceivable that on Λ^1 there is a special need for precisely those means of production that are subject to (possibly relatively large) *positive* price Wicksell effects ($\mathrm{d}p_j/\mathrm{d}r < 0$), whereas on Λ^2 those means of production are necessary which are subject to (possibly relatively large) *negative* price Wicksell effects ($\mathrm{d}p_j/\mathrm{d}r > 0$).[2] The value of the means of production employed on Λ^1, therefore, falls less rapidly than the value of those employed on Λ^2. Consequently, the ordering of rents of these two qualities of land is interchanged: $\rho^1_{(4)} < \rho^2_{(4)}$. Their ordering according to fertility (i.e. profitability) now differs from the ordering of their rents per acre.

This case can be presented graphically. In Fig. 2 we sketch the $\rho(r)$-relationships of two processes of corn production employed on different qualities of land. Process x is employed on the most fertile land Λ^x, process y on land of 'medium' quality, Λ^y. If at an exogenously given wage rate Λ^x is the marginal land, the rate of profit is r_x; if Λ^y is the marginal land, the rate of profit is r_y ($r_x > r_y$). Λ^x yields a rent, as soon as worse land is taken into cultivation and the rate of profit, therefore, falls below r_x. Λ^y yields a rent, as soon as land of a worse quality than Λ^y is taken into cultivation and the rate of profit falls below r_y. While the rate of profit falls, the rents ρ^x and ρ^y rise. It may happen that

$$\left|\frac{\mathrm{d}\rho^y}{\mathrm{d}r}\right| < \left|\frac{\mathrm{d}\rho^x}{\mathrm{d}r}\right|.$$

[1] Cf. Montani, pp. 227–8.

[2] To be able to adduce *negative* as well as *positive* price Wicksell effects we must assume for a moment that the prices of commodities are not expressed in terms of corn quantities, but in terms of some other standard; for all prices that are expressed in terms of corn are subject to *negative* price Wicksell effects (differing only in magnitude), i.e. they fall, when the rate of profits falls. For the term 'price Wicksell effect', which more correctly should be called 'price Ricardo effect', cf. C. F. Ferguson and D. L. Hooks, 'The Wicksell effects in Wicksell and in modern capital theory', *History of Political Economy*, vol. 3, 1971, pp. 353–72, and G. C. Harcourt, *Some Cambridge Controversies in the Theory of Capital*, Cambridge, 1972, pp. 39–46.

The rent per acre yielded by Λ^y, therefore, above r^* in the diagram, is smaller than the rent yielded by Λ^x, and exceeds it at lower rates of profit; i.e. $\rho^y(r)$ and $\rho^x(r)$ intersect. Above r^*, the ordering according to fertility coincides with the ordering of the rents; below r^*, they diverge: the less fertile land yields the higher rent per acre. This contradicts the Ricardian/Marxian point of view that the $\rho^z(r)$ functions do not intersect.[1]

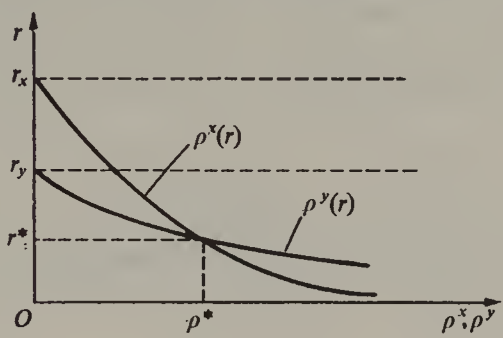


FIG. 2.

III. Differential rent as a form of intensive rent

1. The price system for land of homogeneous quality cultivated with different degrees of intensity

9. In what follows we shall assume that all the land available to society, Λ, is of homogeneous quality and scarce. Thus it is possible that a given demand for corn is satisfied by simultaneously employing two different methods of production on different parts of this land.[2] On account of the

[1] With more complicated cases than the one sketched in Fig. 2 it may even happen that the two $\rho^z(r)$ functions intersect more than once, bringing about the return of the original ordering of the two qualities of land (or, methods of producing corn). This is due to the fact that the price of a product and consequently the value of the capital of an industry (expressed in terms of one of the industrial products) may be subjected successively to *positive, neutral, and negative* price Wicksell effects (see, for example, Sraffa, chap. VI)—a possibility that was noticed neither by Ricardo nor by Marx (see, for example, Ricardo I, pp. 35 and 43, and K. Marx, *Das Kapital* III, vol. 25, p. 213).

[2] Cf. Sraffa, p. 75; Quadrio Curzio, chap. IV; and Montani, pp. 228–40. The first time Ricardo deals with intensively decreasing returns is in his *Notes on Bentham's 'Sur les Prix'*, written in 1810/11 (cf. Ricardo III, pp. 259–341, particularly p. 287). In the *Essay on Profits* (cf. Ricardo IV, pp. 9–41) Ricardo occasionally mentions that the growing demand for corn can be satisfied not only by taking worse-quality or remote areas into cultivation, but alternatively by more intensively cultivating the high-quality or favourably located areas. Which of these two possibilities is preferred is a matter of choice of technique. In the *Principles*, Ricardo writes: 'It often, and, indeed, commonly happens, that before No. 2, 3, 4, or 5, or the inferior lands are cultivated, *capital can be employed more productively on those lands which are already in cultivation.* It may perhaps be found that by doubling the original capital employed on No. 1, though the produce will not be doubled, . . . it may be increased . . ., and that this quantity exceeds what could be obtained by employing the same capital, on land No. 3.—In such case, capital will be preferably employed on the old land, and will equally create a rent; for rent is always the difference between the produce obtained by the employment of two equal quantities of capital and labour.' (Ricardo I, p. 71)

homogeneity of the land, there is only one single rent ρ and consequently one single price of land, p_λ.[1] At first, let us for simplicity assume that corn is nowhere used as a means of production, i.e. that it is a *non-basic* product. The price system for the n industries and the two agricultural processes can be written:

$$\left.\begin{aligned}(1+r)(a_{11}p_1+\ldots+a_{n1}p_n)+wl_1 &= b_1p_1\\ (1+r)(a_{12}p_1+\ldots+a_{n2}p_n)+wl_2 &= b_2p_2\\ \cdots\cdots\cdots&\\ (1+r)(a_{1n}p_1+\ldots+a_{nn}p_n)+wl_n &= b_np_n\end{aligned}\right\}\ \text{(I)}\qquad (4)$$

$$\left.\begin{aligned}(1+r)(a^1_{1g}p_1+\ldots+a^1_{ng}p_n)+wl^1_g+\rho\Lambda^1 &= b^1_gp_g\\ (1+r)(a^2_{1g}p_1+\ldots+a^2_{ng}p_n)+wl^2_g+\rho\Lambda^2 &= b^2_gp_g,\end{aligned}\right\}\ \text{(II)}$$

where Λ^1 and Λ^2 no longer represent land of different fertility, but different acreages of *equal* fertility such that

$$\Lambda \geqslant \Lambda^1+\Lambda^2. \qquad (5)$$

The $n+2$ equations contain $n+4$ unknowns: n prices of industrial products, the price of corn, the rent per acre ρ, r, and w. If we standardize the system by setting one of these prices, say p_1, equal to unity and as before assume the wage rate to be exogenously given, the system is closed.

10. The assumption that corn is a non-basic product has the effect that the rate of profit and the prices of the industrial products are determined solely within the industrial sector, i.e. through (4/I), and serve as data for the agricultural sector. The system is *decomposable*. When we insert the values determined through (4/I) in (4/II) we obtain the price of corn and the rent. We suppose that (4) yields economically meaningful results: positive prices and a positive rent. According to Sraffa, the latter requirement implies that the corn-producing method that yields the higher returns per acre is also characterized by higher costs per unit of output (cwt. of corn). *Costs* here include the value of the means of production, profits, and wages. Sraffa's statement can easily be proved, Let q^i ($i = 1, 2$) designate the costs of producing one cwt. of corn by means of method i:

$$q^i = \frac{(1+r)(a^i_{1g}p_1+\ldots+a^i_{ng}p_n)+wl^i_g}{b^i_g}.$$

(4/II) now can be written as follows:

$$\left.\begin{aligned}q^1+\rho\frac{\Lambda^1}{b^1_g} &= p_g\\ q^2+\rho\frac{\Lambda^2}{b^2_g} &= p_g.\end{aligned}\right\}\qquad (6)$$

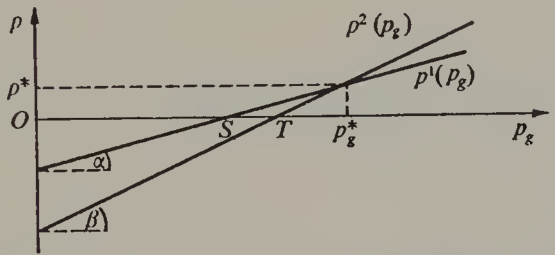
[1] Of course, the price of land is nothing but the capitalized rent: $p_\lambda = \rho r^{-1}$.

Solving (6) for ρ yields:

$$\rho = \frac{q^1 - q^2}{(\Lambda^2/b_g^2) - (\Lambda^1/b_g^1)}.$$

The rent and the price of corn are positive if and only if the higher 'unit cost' of method i ($q^i > q^j$) is connected with lower 'land intensity' ($\Lambda^j/b_g^j > \Lambda^i/b_g^i$), i.e. with higher returns per acre ($b_g^i/\Lambda^i > b_g^j/\Lambda^j$).[1]

In a diagram graphically representing the two $\rho(p_g)$ functions contained in (6) (cf. Fig. 3), the requirements for a positive rent as well as a positive price of corn are fulfilled, if the two curves intersect within the positive quadrant.



$$tg\alpha = \frac{b_g^1}{\Lambda^1}\ ;\ OS = q^1$$

$$tg\beta = \frac{b_g^2}{\Lambda^2}\ ;\ OT = q^2$$

Fig. 3.

2. The variation of the price of corn, the rent per acre and the rate of profit caused by an intensification of cultivation

(a) Corn as a non-basic product

11. We now turn to the question how an intensification of corn production at a given wage rate affects the price of corn, the rent per acre, and the rate of profit. As the answer crucially depends on whether or not corn is a basic product, we shall discuss both cases separately. At first we assume that corn is nowhere used as a means of production. At a given nominal wage, therefore, the prices of the industrial products and consequently the costs per cwt. of corn for all corn-growing methods are known and constant. As before, we assume that all of the h corn-growing methods available to society comply with the dual positivity requirement: $\rho > 0$ and $p_g > 0$. Simultaneously this ensures that the various methods of production yield the higher returns per acre, the more cost-intensively they produce, i.e. the higher their costs per cwt. of corn are. Provided

$$u^1 < u^2 < \dots < u^h, \tag{7}$$

[1] Of course, the costs of inputs, particularly the inputs of produced means of production, again depend on distribution. Hence it is possible that at a given wage rate one method does not only have higher productivity than the other but at the same time has lower unit-costs, whilst at another wage rate it has higher unit-costs. In the first case the second method obviously is inferior to the first and thus has to be substituted by another method which fulfils the above requirement.

where u^i ($i = 1, 2, \ldots, h$) designates the *per acre productivity* of method i,[1] it follows that

$$q^1 < q^2 < \ldots < q^h, \tag{8}$$

where q^i as before represents the unit costs of method i.

12. As long as the demand for corn can be satisfied solely by employing the first method on the available area, there will be no rent. Only when the least productive and least cost-intensive method could no longer satisfy the demand, even if it occupied all the land, does a positive rent appear as a more productive and more cost-intensive method is employed. This

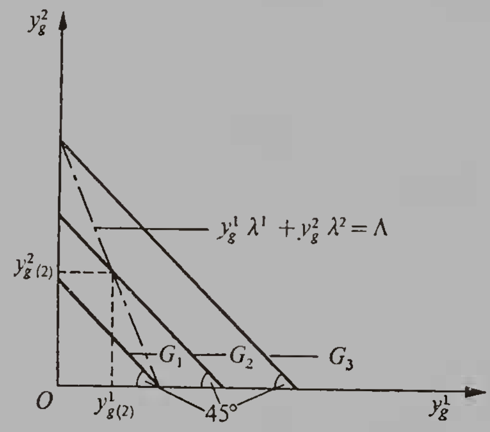


Fig. 4.

reflects the fact that land has become scarce; scarce in so far as corn production can only be expanded at the expense of rising unit costs per cwt. of corn. At constant input prices, the land which is still cultivated by means of the first method obviously yields a positive rent. The rest of the land, where the second method is employed, being of the same quality must permit the payment of the same rent per acre to its owner. In this situation, the total disposable acreage (or a part of it) is cultivated partly by means of the first, partly by means of the second method. The exogenously predetermined demand for corn, G, decides the relative (and absolute) weights of the two methods. Let y_g^1 and y_g^2 be the activity levels of the first two simultaneously employed methods of cultivation ($y_g^1 \geqslant 0$, $y_g^2 \geqslant 0$) and λ^1 and λ^2 the respective land intensities (the reciprocals of the per acre productivities u^1 and u^2). The condition for equilibrium of demand and supply of corn now can be written as follows:

$$y_g^1 + y_g^2 = G. \tag{9}$$

Both activity levels must be compatible with the available agricultural acreage:

$$y_g^1\lambda^1 + y_g^2\lambda^2 \leqslant \Lambda. \tag{10}$$

[1] As we assume constant returns to scale for each method, u^i is constant irrespective of the activity level at which method i is used.

We may discriminate between three cases (cf. Fig. 4): when the demand for corn amounts to G_1, only the first method is employed, i.e. $y_g^1 = G_1$ and $y_g^2 = 0$. An expansion of the corn production beyond G_1 is made possible by successively substituting the more productive method 2 for the less productive method 1; at demand G_2 ($G_1 < G_2 < G_3$), e.g., both methods are employed at activity levels $y_{g(2)}^1$ and $y_{g(2)}^2$ respectively. Finally, at demand G_3, solely the second method is used, i.e. $y_g^2 = G_3$ and $y_g^1 = 0$.

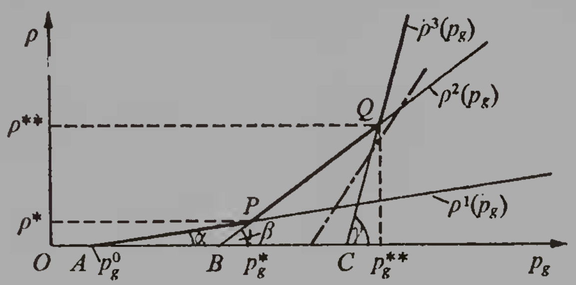


FIG. 5.

In the third as well as in the first case, the agricultural sector consists of one single process. In contrast to the rentless situation $G \leqslant G_1$, the price equation associated with $G = G_3$ contains the rent as an additional variable. So the system as a whole now has three degrees of freedom. Even if we, as usual, exogenously fix the values of two variables (e.g. the wage rate and one of the industrial prices), we are still not able to determine p_g and ρ. All we can say is that the price of corn and the rent per acre may not fall below the values that they take when $G = G_2$. They may, however, exceed these values to a certain extent.[1]

13. When the demand for corn continues to rise, an even more productive method 3 must be employed besides method 2. 'In this way the output may increase continuously, although the methods of production are changed spasmodically.'[2] Since the third method is characterized by higher costs per cwt. of corn, it can be employed only if the price of corn rises. Thus the rent yielded by using the second method increases. The price of corn must rise just so far that a tenant-capitalist who uses the third method can pay the landlord the same rent per acre, and at the same time receives the same rate of profit as a tenant-capitalist who still employs method 2. The new and higher level of p_g and ρ will remain unchanged as long as these two methods coexist, i.e. as long as the third method has not completely displaced the second because of an increasing demand for corn. Again we can present our argument graphically (see Fig. 5).

Those intersections of the process-specific (*not* as in the case of extensive

[1] This will be explained in more detail in the next paragraph. [2] Sraffa, p. 76.

rent, quality-specific) $\rho^i(p_g)$ functions that lie on the boldly typed curve (envelope) determine the equilibrium p_g, ρ combinations in those cases, when two methods, which are next to each other in the ordering described by (7) and (8), are simultaneously employed. The intersection marked P represents the situation when methods 1 and 2 coexist; that one marked Q represents the situation when methods 2 and 3 coexist. The corresponding combinations of the price of corn and the rent per acre are (p_g^*, ρ^*) and (p_g^{**}, ρ^{**}) respectively. In the rentless case, when only the first method is employed, the price of corn amounts to p_g^0.[1]

The indeterminacy of the price of corn and the rent per acre, which comes about whenever there is only one single method employed, is graphically represented by the straight lines of the envelope between the intersections of the $\rho^i(p_g)$ functions. In a situation when only method 2 is employed all the combinations of p_g and ρ lying on $\overline{PQ}$ are solutions of the system. Consequently, in order to close the system, one of the two variables must be assigned a value from this value space.[2]

By means of Fig. 5 we can ascertain what determines the rise in the price of corn and the rent per acre caused by the expansion of production. We know that the method that yields the higher returns per acre also entails the higher unit-costs represented by the intersections of the $\rho^i(p_g)$ functions and the abscissa ($q^1 = \overline{OA}$, $q^2 = \overline{OB}$, $q^3 = \overline{OC}$, and so on). So the rise in the price must at least compensate for the rise in costs caused by a new method coming into use. Moreover, the rise in price and rent depends on the differences in productivity between the various methods, which are expressed by the different slopes of the $\rho^i(p_g)$ functions ($u^1 = \operatorname{tg}\alpha$, $u^2 = \operatorname{tg}\beta$, $u^3 = \operatorname{tg}\gamma$, and so on). The more favourable the ratio between the rise in productivity and the rise in costs turns out to be, i.e., the closer the two intersections of the respective $\rho^i(p_g)$ functions with the abscissa are to each other, and the more the slopes of these two functions differ, the smaller is the rise in p_g and ρ.

(b) Corn as a basic product

14. We shall now remove the unrealistic assumption that corn is nowhere, not even in its own production, employed as a means of production.[3] Corn

[1] The fact that the various methods of growing corn can be ranked according to productivity as well as unit-costs does not mean that all of the methods appearing in this ranking are actually going to be used. This is so because a method may be strictly dominated by the combination of those two methods which are 'framing' it in the ranking of all methods; see, for example, the broken $\rho(r)$ curve in Fig. 5 which is dominated by methods 2 and 3.

[2] The indeterminacy of the two variables at the switching points from one pair of methods to the next (two neighbouring pairs always have one method in common) vanishes in the *continuous* case when the various methods differ only marginally.

[3] That is why the argument in the preceding paragraphs is not applicable to corn production but possibly, for example, to pearl fishery or oyster breeding.

is a *basic product* again. We must, therefore, replace equation system (4) by:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
(1+r)(a_{11}p_1+\ldots+a_{n1}p_n+a_{g1}p_g)+wl_1 = b_1p_1 \\
(1+r)(a_{12}p_1+\ldots+a_{n2}p_n+a_{g2}p_g)+wl_2 = b_2p_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(1+r)(a_{1n}p_1+\ldots+a_{nn}p_n+a_{gn}p_g)+wl_n = b_np_n
\end{array}\right\} \quad (\mathrm{I}) \\
\left.\begin{array}{l}
(1+r)(a^1_{1g}p_1+\ldots+a^1_{ng}p_n+a^1_{gg}p_g)+wl^1_g+\rho\Lambda^1 = b^1_gp_g \\
(1+r)(a^2_{1g}p_1+\ldots+a^2_{ng}p_n+a^2_{gg}p_g)+wl^2_g+\rho\Lambda^2 = b^2_gp_g.
\end{array}\right\} \quad (\mathrm{II})
\end{array}\right\} \quad (11)$$

Again, the system has two degrees of freedom. If we set one of the prices, say p_1, equal to unity and assume the wage rate to be exogenously predetermined the system is closed.

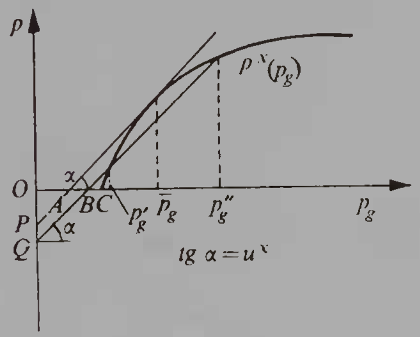


FIG. 6a.

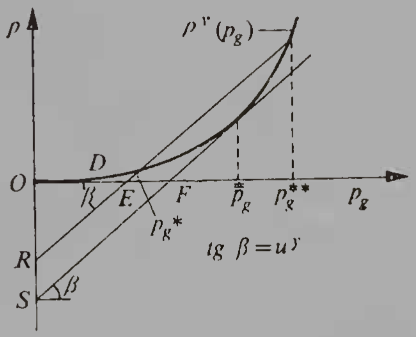


FIG. 6b.

15. As soon as corn directly or indirectly enters into the production of all the $n+1$ commodities, the $\rho^i(p_g)$ functions generally can no longer be linear, because every variation of the price of corn is connected with a change in the rate of profit and in all industrial prices save the one that serves as *numéraire*. When (using the above definitions of u^i and q^i) we reduce the price equations of the h different processes available in the agricultural sector to the general form

$$\rho^i = u^ip_g - u^iq^i(p_g) \qquad (i = 1, 2, \ldots, h), \tag{12}$$

it immediately becomes evident that the costs per cwt. of corn depend on the price of corn itself. Two extremely different cases of the $\rho^i(p_g)$ relationship are sketched in Fig. 6. The $\rho^x(p_g)$ function presents the case where the rent per acre increases degressively while the price of corn rises; the $\rho^y(p_g)$ curve shows a progressively increasing rent per acre.[1] Since the productivity

[1] Since the price of corn rises relatively to all single industrial prices, the wage rate being constant and the rate of profits falling, it also rises relatively to the unit-costs q^i of corn production of any method i. Thus, according to (12), ρ is a monotonically increasing function of p_g. In case of bent $\rho^i(p_g)$ functions (i.e. where corn is a basic product), however, it is no longer valid that methods involving lower productivity will necessarily be replaced by methods with generally higher productivity, For this point see Montani, pp. 239–40.

per acre is a ratio of *physical* quantities independent of the price system, the unit-costs at a given price of corn can be determined by means of a simple graphical device: If, for example, in the former (latter) case the price of corn is p_g'' (p_g^{**}), the straight segment $\overline{OQ}$ ($\overline{OR}$) on the negative branch of the ordinate represents the value of $u^x q^x(p_g'')$ $[u^y q^y(p_g^{**})]$. Now it is true that

$$\operatorname{tg}\alpha = u^x = \frac{\overline{OQ}}{\overline{OB}} = \frac{u^x q^x(p_g'')}{\overline{OB}}, \text{ whence it follows that } \overline{OB} = q^x(p_g'')$$

$$\left[\operatorname{tg}\beta = u^y = \frac{\overline{OR}}{\overline{OE}} = \frac{u^y q^y(p_g^{**})}{\overline{OE}}, \text{ whence it follows that } \overline{OE} = q^y(p_g^{**})\right].$$

When the price of corn rises, the $\rho^x(p_g)$ curve $[\rho^y(p_g)$ curve$]$ at first implies decreasing (increasing) unit-costs; above $\bar{p}_g$ ($\bar{\bar{p}}_g$), however, the unit-costs will increase (decrease). Hence each unit-costs value in the space reaching from $\overline{OA}+\delta q$ to $\overline{OC}$ (from $\overline{OD}$ to $\overline{OF}-\delta q$) corresponds to two different prices of corn [e.g. $q^x = \overline{OB}$ to p_g' and p_g'' ($q^y = \overline{OE}$ to p_g^* and p_g^{**})]. The two cases considered here constitute the basic patterns for more complicated (and presumably more realistic) cases, where the relationship between rent per acre and price of corn may alternatingly display concave and convex segments.

16. We shall now turn to the question how the simultaneous usage of different methods of growing corn affects the values of p_g and ρ. Let us for simplicity assume that there are only two methods having the same structure as the two characteristic cases described above. Obviously the two methods can only be combined without violating the condition of a uniform price of corn and a uniform rent per acre if their respective $\rho^i(p_g)$ curves intersect.[1] Provided that in the rentless situation the $\rho^x(p_g)$ function entails lower starting costs than the $\rho^y(p_g)$ function (i.e. $\overline{OC} < \overline{OD}$), they will intersect only once, otherwise (i.e. $\overline{OC} > \overline{OD}$) twice. Let us immediately turn to the more interesting second case (sec Fig. 7). The two equilibria are P^+ and P^{++}. In accordance with (7) and (8), the method with the higher per acre productivity at the same time entails higher unit-costs. Since the two equilibria imply the same quantitative structure of production and the same maximum corn output, the question of which possibility will be preferred cannot be answered within the model, but only by taking additional criteria into account.

The alternative of P^+ or P^{++} reflects a *distributional conflict* between the landlords on the one hand, and the capitalists and the labourers on the other. The point of view advocated by Ricardo saying that in a *stationary economy* only the conflict between capitalists and labourers matters—'If wages fell,

[1] Of course, the requirements are met already if the two curves are tangent to one another.

profits, *and not rent*, would rise. If wages rose, profits, *and not rent*, would fall'[1]—therefore *needs correction*: At a price of corn p_g^{++} the money rent per acre[2] amounts to ρ^{++} thus exceeding the rent ρ^{+} associated with the lower price of corn p_g^{+}. Since the wage rate expressed in terms of industrial product 1 remains constant as assumed above, at a transition from P^{+} to P^{++} the *real* wage rate of the Ricardian system expressed in terms of corn (cwt.) will fall. In contrast to Ricardo's view presented above, a falling

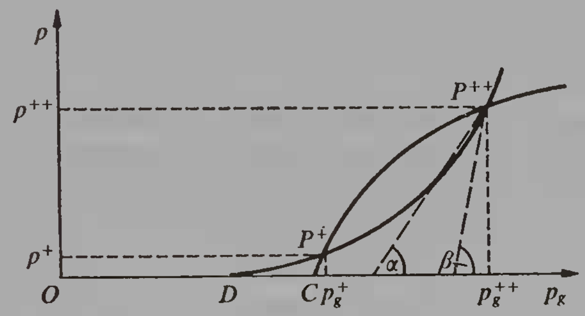


FIG. 7.

real wage rate is compatible with a rising rent per acre. But how about the variation of the rate of profits? According to Ricardo it should be higher at P^{++} than at P^{+} because of the lower corn wage rate, but actually it is lower. This becomes evident when we consider how the higher price of corn is 'absorbed' within the industrial sector. Since all the industries directly or at least indirectly employ corn as a means of production, a rise in the price of corn at first, i.e. *ceteris paribus*, has the effect that they produce at a (virtual) loss. To compensate for this deficit without a reduction in the rate of profits the proceeds of all the industries have to be increased by raising the industrial prices. Higher prices, however, imply higher input costs. A reduction in the rate of profit, therefore, is inevitable.[3]

The distributional conflict is obvious: The situation P^{+} is advantageous both to the workers, providing them with a higher real wage rate (in terms of corn), and to the capitalists, who receive a higher rate of profits. The landlords, however, prefer P^{++} because of the higher rent per acre and the higher corn price. A possible answer to the question raised above could be that the choice between P^{+} or P^{++} depends on the relative power of the two dissimilar parties within the distributional conflict.

[1] Ricardo I, p. 411.

[2] Strictly speaking, the model does not deal with monetary terms, but with values in terms of units of commodity 1.

[3] At the transition from P^{+} to P^{++} some of the industrial prices may fall in relation to p_1, whereas some remain constant, and some rise. In relation to p_g, however, *all* of the prices must fall. Of course, the rise in the price of corn must comply with the requirement of a positive rate of profit.

IV. Conclusion

It has been shown that Sraffa's approach to rent theory, although undoubtedly rooted in the classical tradition, provides us with the foundation of a critique of the latter in several respects. The ranking according to fertility has turned out not to be naturally given but to depend on distribution. With the wage rate (rate of profits) hypothetically varying, the same ranking may return—the reswitching of techniques phenomenon thus is exposed in a world with unproduced means of production. Moreover, it has been demonstrated that the ranking of different qualities of land according to fertility and the ranking according to rent per acre in general do not coincide, as was asserted by Ricardo and Marx. The possibility of multiple solutions to the system of prices, the real wage rate, the rate of profits, and the rent per acre in the case where all the land is of uniform quality (with corn as a basic product) indicates the existence of a further distributional conflict besides the one between capitalists and workers.

Of course, the neo-Ricardian approach to explaining the existence and magnitude of differential rent is not limited to agricultural production. Whenever there is a homogeneous factor of production used at different degrees of intensity or a non-homogeneous factor divided into several qualities, some or all of which are scarce, the neo-Ricardian theory may be applied. Ricardo already points out the further-reaching aspects of his approach:

> If air, water, the elasticity of steam, and the pressure of the atmosphere, were of various qualities; if they could be appropriated, and each quality existed only in moderate abundance, they, as well as land, would afford a rent, as the successive qualities were brought into use.[1]

One example where we can extend our analysis are natural resources, such as land or mineral deposits, equal in quality, but differently located. Ricardo emphasizes that this kind of rent is to be regarded as a sub-species of rent due to extensively diminishing returns.[2] In his *Essay on Profits* he supposes land to be of uniform fertility, but non-homogeneous because of different distances to the centre of demand for corn. In this case rent is equal to the difference between the costs of transportation from marginal and from favourably located land. If the wage rate in terms of corn remains constant, 'profits are regulated by the difficulty or facility of procuring food'.[3] The emphasis lies on the word *procuring* in contrast to *producing*.

[1] Ricardo I, p. 75.

[2] See, for example, Ricardo I, pp. 70 and 72; cf. also K. Marx, *Das Kapital* III, vol. 25, pp. 663–4.

[3] Ricardo IV, p. 13. For the Ricardian theory of the rate of profits in the simple 'corn model' see, for example, P. D. Groenewegen, 'Three notes on Ricardo's theory of value and distribution', *Australian Economic Papers*, vol. 11, 1972, pp. 53–64, and S. Hollander, 'Ricardo's analysis of the profit rate, 1813–15', *Economica*, vol. 40, 1973, pp. 260–82.

Indeed, in this context the difficulty not in growing, but in procuring corn—including the transportation to the consumers—plays the decisive role in determining the rate of profit and the rent per acre.[1] Obviously, as far as rent is concerned, this way of reasoning applies likewise to mineral deposits which are found and exploited at different distances to the centres of manufacturing.

Machines of an obsolete type which are still in use as means of production are similar to land of worse quality. Implicitly, Ricardo already deals with this case in a gloss on Adam Smith's opinion on rent. If rent were advantageous to society, as Smith maintains, it would likewise be desirable

> that, every year, the machinery newly constructed should be less efficient than the old, as that would undoubtedly give a greater exchangeable value to the goods manufactured, not only by that machinery but by all the other machinery in the kingdom; and a rent would be paid to all those who possessed the most productive machinery.[2]

In contrast to the usual vintage capital models, here Ricardo's vintages are based on technical retrogression instead of technical progress—an analogy to the resort from the best to worse and worse qualities of land. Apart from this peculiarity, vintages in fact constitute another application of the neo-Ricardian theory by causing the industrial *quasi-rents* of which Sraffa writes: 'The quasi-rent (if we may apply Marshall's term in a more restricted sense than he gave it) which is received for those fixed capital items which, having been in active use in the past, have now been superseded but are worth employing for what they can get, is determined precisely in the same way as the rent of land.'[3]

If technical progress embodied in new methods of production is not spread throughout the whole economy but monopolized by one or a few producers (e.g. because of patents or differently 'dynamic' entrepreneurs), allowing them to produce more cheaply than their competitors and to pocket extra profits, again neo-Ricardian rent theory can be applied.

Without too great a stretch of the imagination, outside agriculture intensively diminishing returns are found, for example, in the supply of

[1] It should be noticed that the identity of the methods of production employed on equally fertile land as well as the differing costs of transportation have an important consequence: In contrast to rent accruing as a result of heterogeneous qualities of land, rent due to location is characterized by the fact that it is the higher the more favourably the land is situated. The ordering of land according to its location, therefore, *always* coincides with the ordering according to rents per acre. Here the view held by Ricardo and Marx concerning the rent on land of different qualities is confirmed (see paragraph 6 of this paper).

[2] Ricardo I, p. 75. One of the passages in *The Wealth of Nations* that Ricardo attacks most violently reads: 'rent may be considered as the produce of those powers of nature, the use of which the landlord lends to the farmer. It is greater or smaller according to the extent of those powers' (A. Smith, vol. i, pp. 343–4).

[3] Sraffa, p. 78.

housing and offices in the cities. The scarcity of land is reflected by the different heights of the buildings, which may be regarded as different methods of production employed side by side. If there was no scarcity, only the cheapest method would be used on the best-located areas in town and there could be no rent.

Institut für Theoretische Volkswirtschaftslehre der Universität Kiel

[14]

A NEO-RICARDIAN ANALYSIS OF INTERNATIONAL TRADE*

I. INTRODUCTION

Traditionally, the Ricardian theory of international trade is thought of in terms of a simple Labour Theory of Value, in which prices are proportional to labour inputs. RICARDO himself recognised that the existence of a positive rate of profit could make prices diverge from their direct labour costs but thought the difference was insignificant[1]. Among post-Ricardians, TAUSSIG is one of the few who explicitly recognised that profit on capital could affect the pattern of trade, but he too tended to minimise its importance[2]. More recently, however, differences in the no-trade pattern of income distribution (which, when countries use the same techniques implies a difference in their no-trade rates of profit) has been suggested as a basis for international trade[3].

The purpose of this paper is to develop a general framework for a neo-Ricardian approach to the analysis of trade. Whilst, to some extent, it is pedagogic[4], it does also try to reach some new conclusions in its consideration of trade between countries with different techniques, between countries having available a multiplicity of techniques, and between countries having planned economies. But its chief purpose is to emphasise that the analysis of international trade can be viewed essentially as a particular problem of the choice of technique.

* I should very much like to thank IAN STEEDMAN for suggesting many improvements in this paper. I also wish to thank J.S. METCALFE and W. PETERS for their helpful comments. Of course, errors are entirely my responsibility.

1. See P. SRAFFA (Ed.), *The Works and Correspondence of David Ricardo*, Vol. I, p. 36, Cambridge University Press, London 1951.

2. F. W. TAUSSIG, *International Trade*, Ch. 7, Macmillan, New York 1927.

3. See IAN STEEDMAN and J.S. METCALFE, 'The Non-Substitution Theorem and International Trade Theory', *Australian Economic Papers*, Vol. 12 (1973), pp. 267–269.

4. In as much as it is an elaboration of the suggestion made by STEEDMAN and METCALFE, *ibid.*

L. MAINWARING

The effect of trade is analysed by *comparing* the equilibrium situation in a pair of economies which do not practice trade with a pair that does. The economies *within* a pair may have the same or different technical possibilities, but possibilities are the same as *between* pairs. In other words, although we are talking of trade between two countries A and B, conceptually there are four economies involved: A^0, B^0 and A^T, B^T, the first two being non-trading economies, the second two being their trading counterparts. The 'choice' between, for example, the equilibrium situations A^0 and A^T is made by comparing the obtainable combinations of the real wage and rate of profit. The dual relationship between consumption per worker and the rate of growth is used to assess the gains from trade.

II. THE WAGE, RATE OF PROFIT AND RELATIVE PRICES IN THE CLOSED ECONOMY

We use a circulating capital model in which production takes place in self-contained periods, the wage being paid at the end of each period. There are two commodities (1 and 2) which are produced by means of labour and the same two commodities: a_j is the input of labour into a unit of j; a_{ij} the input of commodity i into a unit of j. Writing p $(=p_1/p_2)$ as the price ratio, and expressing the wage, w, in terms of good 2, we have the following price equations:

$$p = (1+r)(a_{11}p + a_{21}) + w\,a_1 \tag{1}$$

$$1 = (1+r)(a_{12}p + a_{22}) + w\,a_2 \tag{2}$$

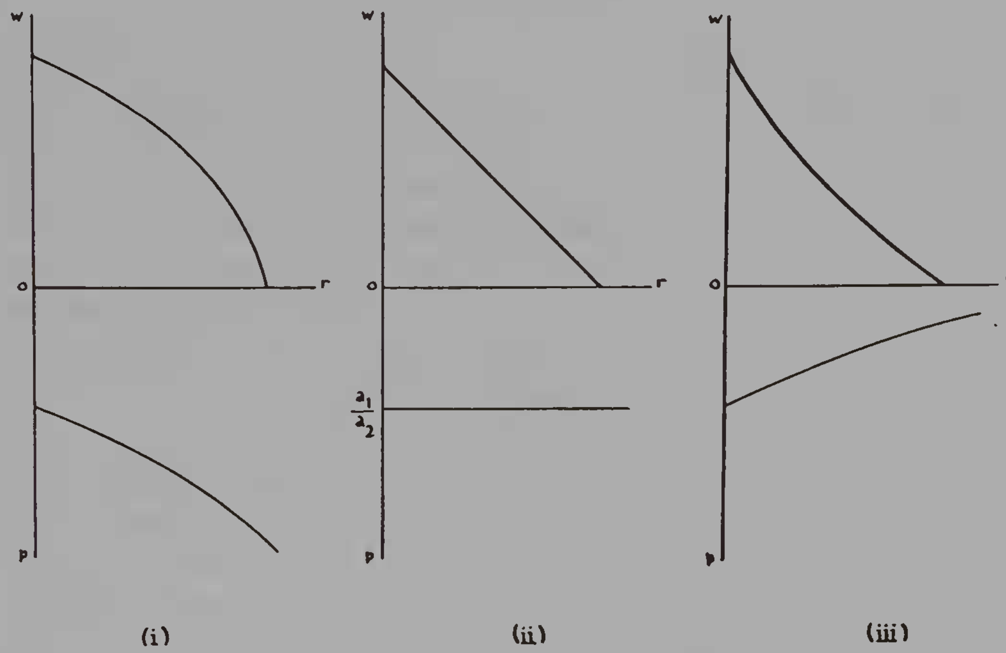
Competition ensures the equality of r (the rate of profit), w and p throughout the economy. (1) and (2) are readily solved to obtain a wage-profit $(w-r)$ relationship:

$$w = \frac{\det(I - vA)}{(1 - v\,a_{11})\,a_2 + a_1\,a_{12}\,v} \tag{3}$$

(where $v = 1 + r$; $A = \|a_{ij}\|$; and I is the identity matrix); and relative prices as a function of r:

A NEO-RICARDIAN ANALYSIS OF INTERNATIONAL TRADE

Figure 1



$$p = \frac{(1 - a_{22}\,v)\,a_1 + a_{21}\,a_2\,v}{(1 - a_{11}\,v)\,a_2 + a_1\,a_{12}\,v} \tag{4}$$

Differentiating (4) with respect to v, we find that

$$dp/dv \gtreqless 0 \quad \text{as} \quad (a_{11}p + a_{21})/a_1 \gtreqless (a_{12}p + a_{22})/a_2 \tag{5}$$

The bracketed terms are the values of commodity inputs into a unit of each process, and the denominators the labour requirements per unit. When these 'capital': direct labour ratios are the same the processes have what MARX called 'equal organic compositions of capital'. In this case relative prices do not vary with r, but generally they are a (monotonic) increasing or decreasing function. By taking the second derivative of w with respect to v in (3), it can be shown that each of these possibilities is associated with a particular shape of the $w-r$ frontier. When $dp/dv > 0$, the $w-r$ frontier is concave to the origin; when $dp/dv = 0$, it is a straight line; and when $dp/dv < 0$ it is convex to the origin (see *Fig. 1*).

L. MAINWARING

III. THE OPEN ECONOMY

The technique represented by the no-trade economy consists, of necessity, of the activities for producing both commodities – of necessity because in the no-trade economy any one activity is not sustainable on its own. But with trade it is possible to have a single activity supported by imported inputs of the other commodity. So the possibility of trade presents two additional techniques, each consisting of a single activity. The problem of technical choice is often presented by superimposing the wage-profit frontiers for each technique to form an outer envelope. Then for any given r, say, the technique that is chosen under competitive conditions is the one which allows the maximum w, *i.e.*, the one on the envelope. In order to approach the analysis in this way it is necessary to derive the wage-profit frontiers for the with-trade techniques. We continue to denote the wage-profit frontier for the no-trade economy by $w-r$; the frontiers for the with-trade techniques will be denoted by $(w-r)_1$ and $(w-r)_2$. For good 1,

$$w=[p-v(a_{11}p+a_{21})]/a_1 \tag{6}$$

A straight line of slope $-(a_{11}p+a_{21})/a_1$. For good 2,

$$w=[1-v(a_{12}p+a_{22})]/a_2 \tag{7}$$

a straight line of slope $-(a_{12}p+a_{22})/a_2$.

For each good there will be a family of these straight lines, each member corresponding to a different price ratio, and thus having a different slope from every other member. There are two points to note:

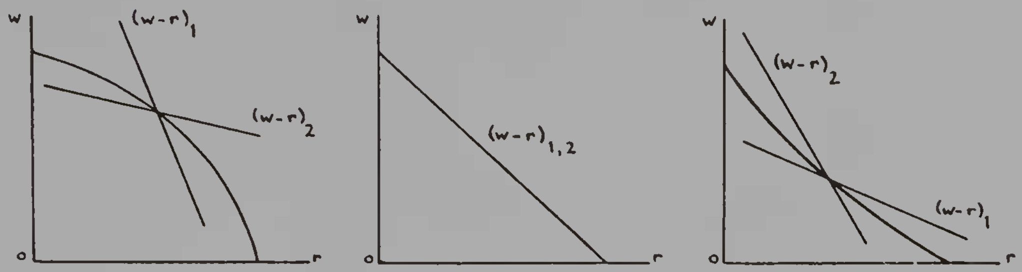
(i) From (5) it follows that when $dp/dv>0$, the slope of $(w-r)_1$ is greater than that of $(w-r)_2$, and *vice versa* when $dp/dv<0$.

(ii) The $(w-r)_i$ frontiers bracket the $w-r$ frontier in a particular manner[5]. Differentiating (6) and (7) with respect to v gives

5. The $w-r$ frontier can be considered as the locus of the $(w-r)_i$ intersections. From (1) and (2) we can obtain expressions for relative prices. These are simply rearrangements of the $(w-r)_i$ expressions; equating and solving for w yields (3).

A NEO-RICARDIAN ANALYSIS OF INTERNATIONAL TRADE

Figure 2



(i) (ii) (iii)

$$\frac{dw}{dv} = \frac{1}{a_1} \cdot (1 - a_{11}\, v) \frac{dp}{dv} + s_1$$

and

$$\frac{dw}{dv} = s_2 - \frac{a_{12}}{a_2} \cdot \frac{dp}{dv} \cdot v$$

where s_1 and s_2 are the slopes of (6) and (7). It follows that if $dp/dv > 0$, then $s_2 > dw/dv > s_1$, or since all slopes are negative we have

$$\begin{array}{lll} dp/dv > 0 & \text{when} & |s_2| < |dw/dv| < |s_1| \\ dp/dv = 0 & \text{when} & |s_2| = |dw/dv| = |s_1| \\ dp/dv < 0 & \text{when} & |s_2| > |dw/dv| > |s_1| \end{array}$$

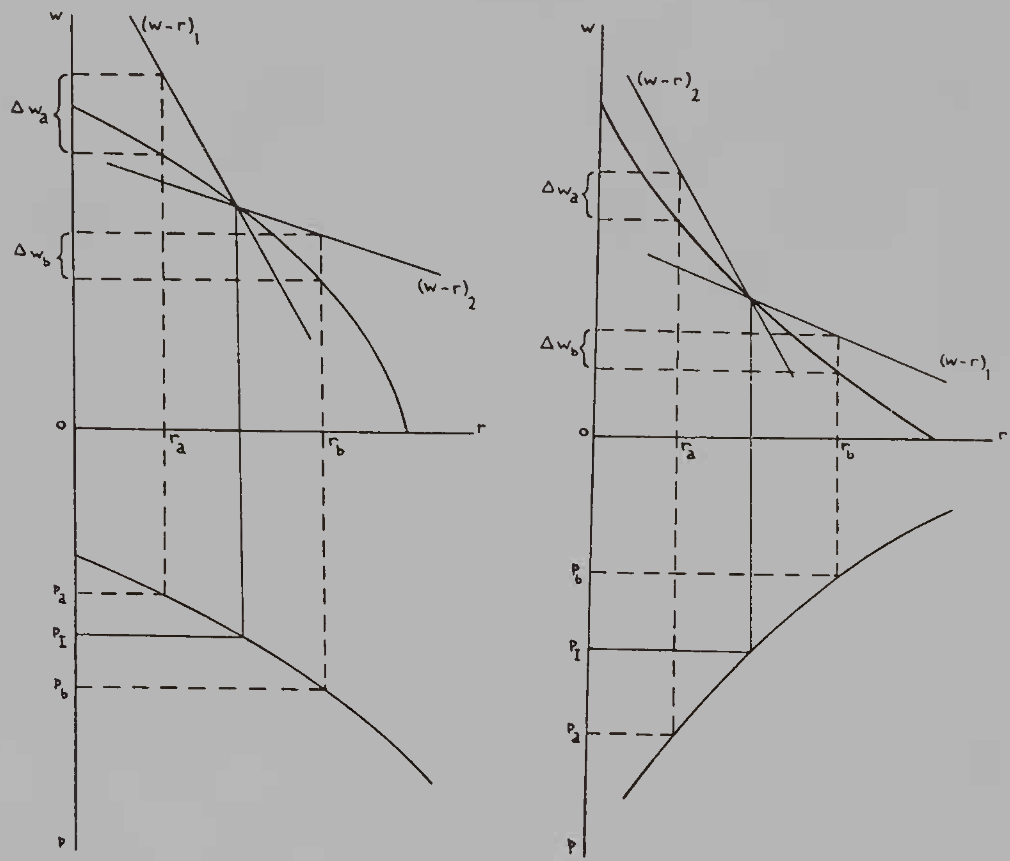
The possible configurations are shown in *Fig. 2.*

IV. FOREIGN TRADE

We are now in a position to introduce trading possibilities between two countries, *A* and *B*. Our only interest is in the comparison of positions of long-run dynamic equilibrium. We assume that there are no impediments to trade, nor any reasons why specialisation should not be realised in either country[6]. We also assume that, for

6. There are technical constraints to specialisation: if one country is very small relative to the other it may not be able to provide all the inputs required in specialised production. In this event the larger country will not be able to specialise.

Figure 3



(i) (ii)

each country, the rate of profit is the same in no- and with-trade equilibria. (We take this case merely for illustration.) If the 'motive' for trade is the attainment of a superior $w-r$ combination, it is seen that such a combination is attained when each country specialises in the commodity which is relatively cheapest to produce.

To begin, suppose that A and B *have the same techniques*, but different rates of profit. In *Fig. 3 (i)*, A and B have the same $w-r$ and $p-r$ functions. Suppose $r_a < r_b$, and hence $p_a < p_b$. The figure shows $w-r$ trade-offs drawn for goods 1 and 2, separately, at some intermediate set of prices p_I. It can be seen that if A specialises in the production and export of good 1 it can obtain a higher real wage with trade, at the rate of profit r_a. Similarly, B has a higher wage through special-

A NEO-RICARDIAN ANALYSIS OF INTERNATIONAL TRADE

isation in good 2. Obviously, if the wage were held constant in the comparison each country would obtain a higher rate of profit. Note that if p_I were equal to either of the no-trade price ratios, the country concerned would be indifferent to trade and would not be induced to specialise. *Fig. 3 (ii)* shows that the pattern of specialisation would be reversed when $dp/dv < 0$. It is evident from *Fig. 2 (ii)* that when $dp/dv = 0$, then no matter how different the profit rates, trade does not allow the attainment of a superior $w - r$ combination.

V. THE GAINS FROM TRADE

Writing gross output of good i per worker (p. w.) as x_i, total consumption (wage plus capitalist consumption) p. w. as e_i, and the rate of growth as g, we have

$$x_1 = (1 + g)(a_{11} x_1 + a_{12} x_2) + e_1 \tag{8}$$

$$x_2 = (1 + g)(a_{21} x_1 + a_{22} x_2) + e_2 \tag{9}$$

Assume, for simplicity, that all consumption is in the form of good 2[7]. Then setting $e_1 = 0$ and solving (8) and (9) for e_2 gives

$$e_2 = c = \frac{\det(I - zA)}{(1 - z a_{11}) a_2 + a_1 a_{12} z} \tag{10}$$

where c is the consumption p. w. (of good 2) and $z = 1 + g$. Equation (10) describes the consumption-growth frontier which is the exact dual of equation (3). Thus the $w - r$ and $c - g$ frontiers coincide. Letting s be the proportion of profits which is saved (we assume that workers do not save) the rate of growth and rate of profit have the familiar relationship $g = s \cdot r$, in long-run steady growth. We assume that $0 \leqq s \leqq 1$, so that $0 \leqq g \leqq r$.

7. This assumption is overly strong. All we require for the following analysis is that the proportions in which the two commodities enter the wage are the same as the proportions in which they enter capitalist consumption. This ensures that the wage and consumption p. w. can be measured in terms of the same commodity bundle. The complete relaxation of this restriction on capitalist consumption would complicate the analysis without substantially changing its conclusions.

Suppose that, with trade, countries specialise fully in production. We can construct $(c-g)_i$ frontiers for each commodity (analogous to the $(w-r)_i$ frontiers). Consider process 2. In value terms, consumption p. w. is equal to gross output p. w. minus the value of inputs required for next period's production. We have everything in the form of good 2 except $a_{12}x_2$ which must be obtained from imports. It can be converted into an amount of good 2 by exchange at the ruling international price ratio, p_I; then

$$c = x_2 - z(p\,a_{12}x_2 + a_{22}x_2)$$

We thus have an expression relating consumption p. w. as a homogeneous physical quantity, and the rate of growth. This procedure is valid so long as one bundle of goods can be transformed into another bundle by means of exchange at the given price ratio. Clearly it is not valid (in general) in the non-specialised or no-trade economy where net output and consumption combinations are transformed according to a technical relationship. Since, for each process operated on its own, $x_i = 1/a_i$ we can write

$$c = [1 - z(p\,a_{12} + a_{22})]/a_2 \tag{11}$$

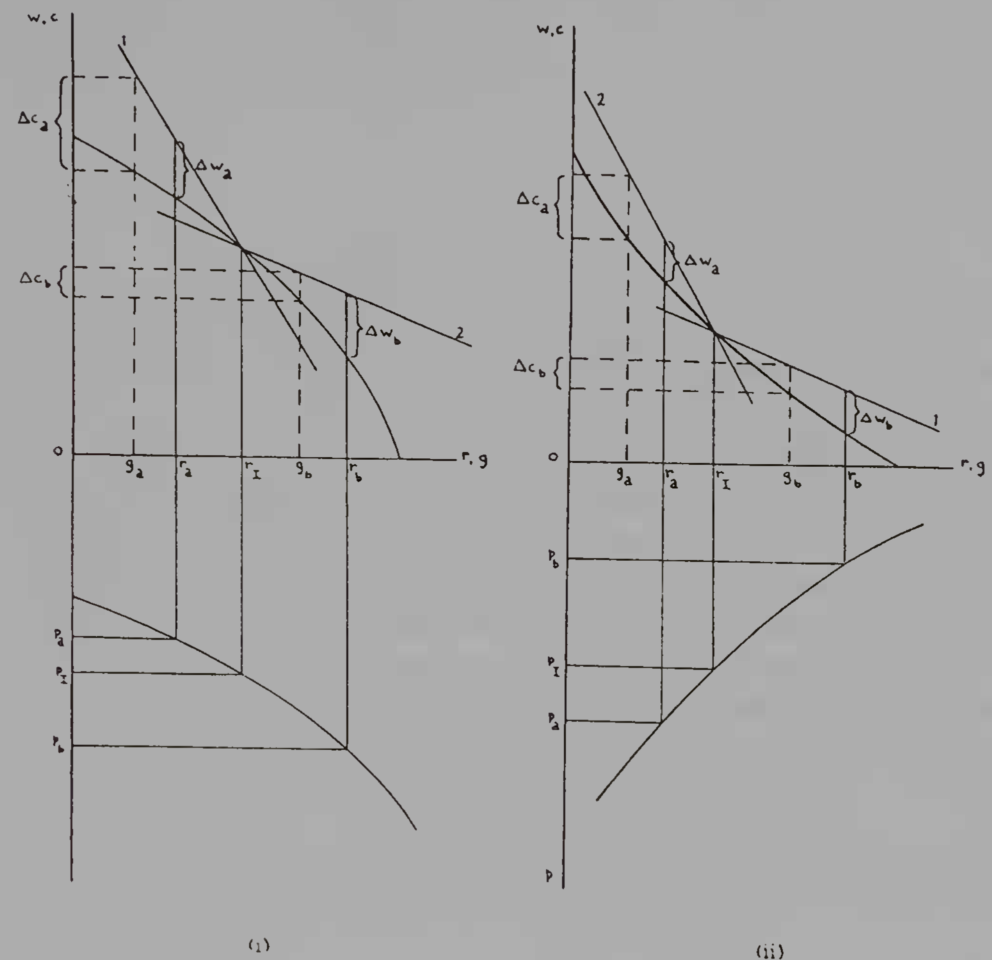
Similarly, for process 1, we have

$$\begin{aligned} c &= p\,x_1 - z(p\,a_{11}x_1 + a_{21}x_1) \\ &= [p - z(p\,a_{11} + a_{21})]/a_1 \end{aligned} \tag{12}$$

Comparison with (6) and (7) shows that the $c-g$ and $w-r$ trade-offs for single processes are identical.

Suppose that, for each country, the rate of profit is the same in no- and with-trade equilibria. In *Fig. 4*, the curved lines are the $w-r$ and $c-g$ frontiers, first for the case $dp/dv > 0$, then for $dp/dv < 0$. The straight lines are the $(w-r)_i$ and $(c-g)_i$ frontiers for the set of prices p_I. First look at *Fig. 4 (i)*. With trade both countries have higher real wages (the differences denoted by Δw_a and Δw_b). In A, the difference in consumption p. w. is larger than the difference in wages

Figure 4



so that capitalist consumption p.w. must be greater with trade. In B, total consumption p.w. is greater with trade *if* $g_b > r_I$ (where r_I is that rate profit associated with a price ratio p_I) *and less if* $g_b < r_I$. But even in the first case $\Delta w_b > \Delta c_b$ so that capitalist consumption p.w. is less.

Now consider *Fig. 4 (ii)*. Again both countries have greater real wages with trade. In A, total consumption p.w. is also greater, but capitalist consumption p.w. may be greater or less, depending on the slope of $(c-g)_2$ relative to that of the $c-g$ frontier. The lower is p_I relative to p_a the more likely it is to be less, since the slope of

L. MAINWARING

$(c-g)_2$ is then smaller. In B, total consumption p.w. with trade is greater if $g_b > r_I$ and less if $g_b < r_I$. In the first case, capitalist consumption p.w. may be greater with trade if p_I is large relative to p_b, for then the slope of $(c-g)_1$ is large relative to the slope of the $c-g$ frontier[8, 9].

VI. COUNTRIES HAVING DIFFERENT TECHNIQUES

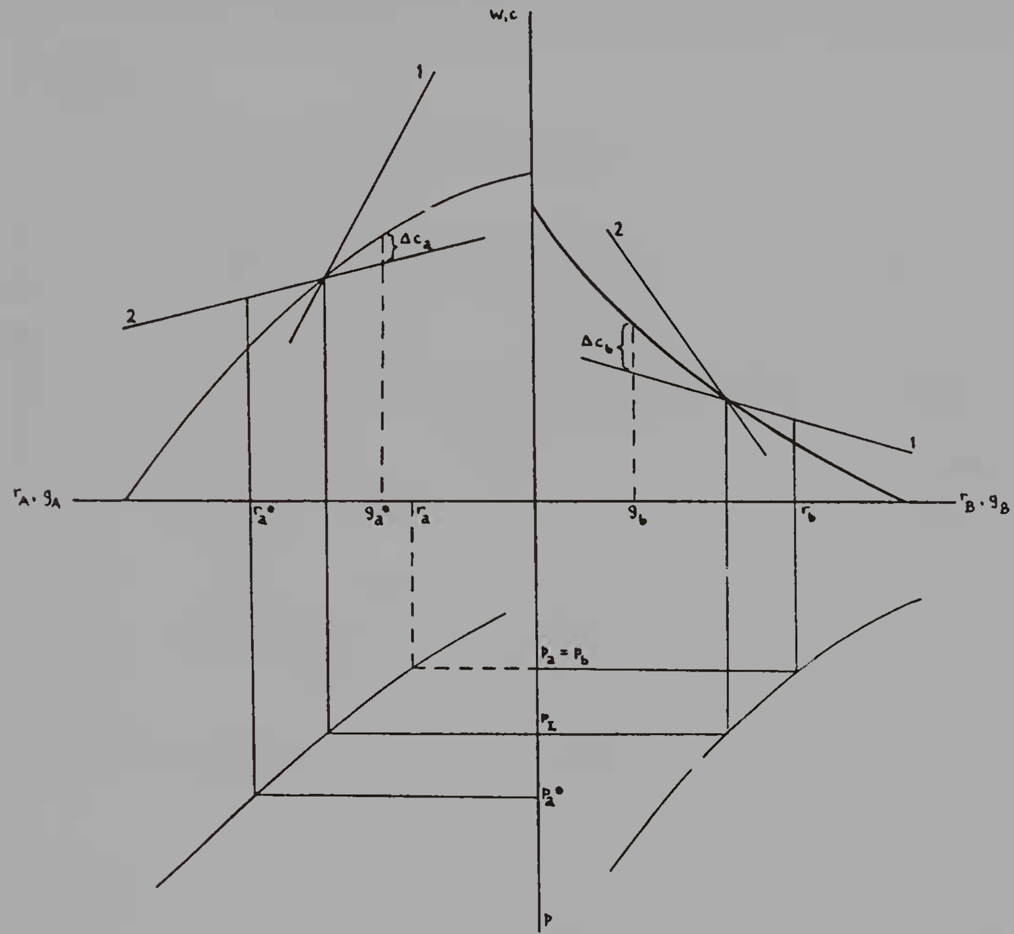
Trade between countries with different techniques can be represented in the back-to-back diagram of *Fig. 5*. On the left are the $w-r$, $c-g$ and $p-r$ functions for A; on the right these for B. The textbook approach to Ricardian trade theory assumes either that rates of profit are zero, or that each country's technology is of the 'equal organic composition of capital' type. Thus relative prices are determined purely by technical coefficients of production. *Fig. 5* brings out the possibility that with more general techniques positive rates of profit might offset or even reverse that pattern of relative prices implied by technical differences when rates of profit are zero[10]. They can be neutralised by choosing in each country rates of profit which correspond to the same relative prices (*e.g.* r_a and r_b); they can be reversed by choosing a rate of profit in A which corresponds to a relative price which is greater than that for r_b (*e.g.* r_a^*). In the latter event it can easily be seen that *both* countries may lose from trade in the sense of having lower consumption possibilities at their respective rates of growth ($\Delta c_a, \Delta c_b < 0$).

8. Similar conclusions regarding the overall p.w. consumption set may be found in a paper by STEEDMAN and METCALFE, in which two consumption commodities are produced in their own integrated sectors: IAN STEEDMAN and J. S. METCALFE, 'The Golden Rule and the Gain from Trade', in the *Proceedings of the Nice Conference on Non-Neoclassical Economics*, Paris, forthcoming. Given that the comparison of consumption–growth possibilities is the appropriate way to measure the gains from trade in the present analysis, these conclusions are in marked contrast to those of static neo-classical analysis.

9. Note that the variation in the capitalist 'gain' from trade with the capitalist savings ratio, in any country, can also be inferred from the diagrams, and will depend on the particular circumstances of that country's equilibrium in trade.

10. It can be seen by inspection, however, that this possibility only arises when the $w-r$ frontier is convex in one country and concave in the other.

Figure 5



VII. MANY TECHNIQUES

Suppose that more than one technique is available to each country in the no-trade economy. The chosen technique at a given rate of profit is the one which yields the highest wage at that rate. So far as the analysis of trade in two commodities is concerned the existence of many techniques makes no significant difference, for once the rate of profit is given the chosen no-trade technique is known and so are relative prices. Although other techniques are available they come into consideration only at other rates of profit. Since we are taking r as given we can proceed as if the chosen technique is the only one. It follows that if two countries have the same technologies (*i.e.*, the same set of available techniques) but their rates of profit are such that the chosen techniques are different, it is possible that both countries may lose from trade[11].

VIII. THE TRADE CRITERION IN A PLANNED ECONOMY

Imagine an economy with a single technique which is planned by a central authority. As before, relationships are defined between the real wage, relative prices and the ratio of the value of net output, less wages, to the value of commodity inputs, which we continue to call the rate of profit[12]. Assume: that all consumption (and hence the wage) is in the form of good 2, that the workforce is growing at a fixed rate n; and that the planner has full knowledge of the technical coefficients of production.

In a *fully* centralised economy the planner 'is a monopsonist in the labour market and fixes the wage rate w, to which labour supply is inelastic'[13]. His functions are twofold: (i) to maintain full employment, and (ii) to maximise p.w. consumption at a given rate of growth. In order to fulfil his first obligation, the planner must ensure that the economy is growing at a rate $g=n$, which is thus a con-

11. It may be noted that one way of ensuring that both countries gain from trade is to define each technique in the available set to be of the 'equal organic compositions of capital' type. Such a set would define a convex $w-r$ envelope (Factor Price Frontier). Countries having the same set of techniques could still trade provided the techniques actually in use were different. (With an infinite number of techniques this simply requires different rates of profit.) It is essentially this assumption that SAMUELSON uses to construct his Surrogate Production Function. *In this sense* the neo-classical theory of international trade may be considered as a special case of the present analysis. See P. A. SAMUELSON, 'Parable and Realism in Capital Theory: the Surrogate Production Function', *Review of Economic Studies*, Vol. 29 (1962), pp. 193–206.

12. Since neo-Ricardian theory by-passes the so-called Transformation Problem we are clearly assuming that the planners formulate commodity prices directly from our price equations rather than via labour values (see ALFRED MEDIO, 'Profits and Surplus Value: Appearance and Reality in Capitalist Production', in: E. K. HUNT and J. G. SCHWARTZ, *A Critique of Economic Theory*, pp. 323–326, Penguin, Harmondsworth 1972).

13. D. M. NUTI, 'Capitalism, Socialism and Steady Growth', *Economic Journal*, Vol. 80 (1970), pp. 32–57.

straint. The level of the real wage must be such that the consequent rate of profit is at least as great as the given growth rate, or the investment required to occupy the workforce in future periods cannot be sustained.

Whether there will be any difference between the trade criterion of planned and free market economies depends on what happens to the surplus product, all of which is appropriated by the state. In the theoretical literature assumptions differ as to how the state disposes of the surplus. According to PASINETTI,

> 'the state, as such, cannot consume: consumption can be carried out only by individuals. Therefore, if any amount of the national product is not distributed to the members of the community [...] that amount is *ipso facto* saved. This means that the parameter s becomes unity [...][14].'

In this event, $g=r$ and $c=w$, so that maximising w at given r (the capitalist criterion) also maximises c at given g. On the other hand, NUTI assumes that the planner re-invests some proportion, s, $(0 \leq s \leq 1)$ of the surplus product, the rest being distributed by the state either as collective consumption or as a wage subsidy[15].

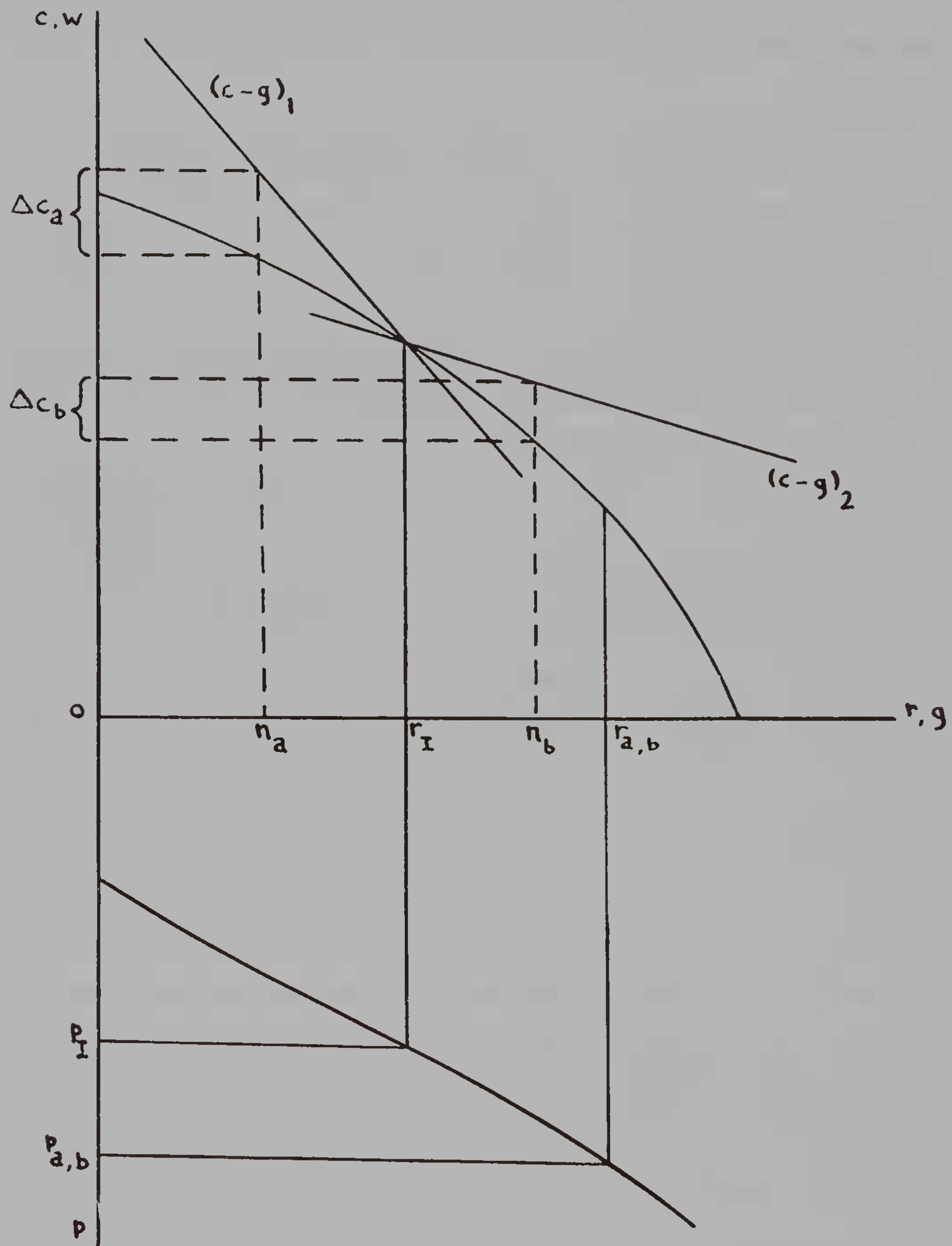
If we accept NUTI's assumptions, and take $s<1$, the trade criterion will be different. Consider the possibility of specialised trade between two planned economies (assuming that their techniques are identical). For each country there is a problem of technical choice, each with-trade technique consisting of a single process, and the desired process is the one giving the highest p.w. consumption at the given rate of growth. In terms of a trade criterion the with-trade situation is superior (inferior) if p.w. consumption, at $g=n$, is greater (smaller) than it is with no trade. It follows that differences in no-trade price ratios are *not* a satisfactory criterion for trade, for maximising w at given r does not imply maximising c at given g. We have already seen that a process chosen according to this criterion might have a p.w. consumption (at given g) inferior to its no-trade level.

But there does exist some trading position which is mutually ad-

14. L. PASINETTI, 'Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth', *Review of Economic Studies*, Vol. 29 (1962), pp. 267–279.

15. NUTI, *op. cit.* (unlike NUTI, we assume that workers' savings are zero and that the state imposes no taxes on wages). Presumably, a subsidy would here be defined as that part of workers' consumption which is an addition to that implied by the chosen price ratio.

Figure 6



vantageous, namely, when international prices p_I are such that the associated rate of profit, r_I, has a value such that $n_a < r_I < n_b$ *(Fig. 6)*. Thus, when two planned economies have the same single technique production possibilities, gainful trade is possible if their given rates of growth are different. It is, therefore, possible for gainful trade to take place when the no-trade price ratios are the same, so long as $s_a \neq s_b$. Conversely, if $r_a \neq r_b$, but $n_a = n_b$, mutually gainful trade is not possible.

If the technique in A is different from that in B, the trade situation can be illustrated by means of a back-to-back diagram, gainful trade being possible if those price ratios which would exist if the rates of profit were equal to the respective rates of growth are different.

What happens to the criterion when the economy is not completely centralised?

'Under decentralised socialism physical productive assets belong to state firms. Firms have access to a perfectly competitive labour market, and have infinite power of borrowing and lending [capital] from and to the State [...]. They appropriate current output and pay wages and interest out of it. Among the production techniques available, they select the technique maximising the present value of their assets at the ruling rate of interest. The socialist planner will still wish the technique maximising consumption per head to be chosen, but the only way he can affect technical choice is by choosing the rate of interest r, which is the basis of the decisions of State managers[16].'

Firms which are attempting to maximise their assets at a given r, trade according to the capitalist criterion. To ensure that this is consistent with the maximisation of consumption at the given growth rate it is sufficient that the planner sets the rate of profit equal to the rate of growth – NUTI's version of the Golden Rule.

IX. CONCLUSION

Under capitalism and decentralised socialism, firms approach the problem of trade as they would any other problem of technical choice: according to the profit maximising criterion. This is implied by trade according to differences in no-trade relative prices. Under capitalism this approach could well result in one or both of the participants realising a poorer consumption p.w.-rate of growth combination than would be the case under autarky. When both countries practice decentralised socialism the planners' control of the rates of profit can assure that both countries always gain from trade.

16. NUTI, *op. cit.*

L. MAINWARING

In fully planned economies the direct purpose of trade is the improvement in $c-g$ possibilities and the trade criterion is not implied by differences in relative prices.

The method of analysis presented in this paper can, in principle, be extended in a number of directions, since the consequences for trade of various actions can be deduced from the resulting modifications in the $w-r$ and $c-g$ frontiers[17].

Reading University L. MAINWARING

SUMMARY

Countries which have the same no-trade techniques may still profitably enter into trade, providing that their rates of profit differ, for this implies differences in relative prices. The analysis is conducted by comparing the wage-profit frontiers appropriate to the production activities in the autarkic and trading equilibria. The motive for trade can be said to be the pursuit of a superior rate of profit-real wage combination. The 'gains from trade' are assessed by comparing the autarky and trade consumption-growth frontiers. If countries have the same single technique, one country will always gain from trade while the other may lose. If countries have different techniques then both may lose. When each country can choose from a set of techniques then, even if the sets are the same, both may still lose. These possibilities could be avoided in planned economies, whether centralised or decentralised, by ensuring that the maximisation of consumption at a given rate of growth is the objective of trade.

ZUSAMMENFASSUNG

Lander mit gleichen Vorhandelstechniken können noch immer vorteilhaft Handelsbeziehungen aufnehmen. Voraussetzung dazu ist ein Unterschied in ihren Profitraten, da dieser auch Unterschiede in den relativen Preisen impliziert. Die Analyse vergleicht die Lohn-Zins-Grenzen, die den Produktionsaktivitäten im autarken und im Aussenhandelsgleichgewicht entsprechen. Das Motiv des Aussenhandels kann in der Erzielung einer höheren Profitraten-Reallohn-Kombi-

17. I have in mind the effects of tariffs and domestic taxes. Also, in a multi-commodity analysis the resolution of the $w-r$ frontiers into their component $(w-r)_i$ trade-offs allows for a detailed examination of the behaviour of prices as the rate of profit varies which proves of use in studying the possibility of 'factor price' equalisation through trade. This, however, is the subject of another paper.

A NEO-RICARDIAN ANALYSIS OF INTERNATIONAL TRADE

nation gesehen werden. Die Schätzung der Gewinne aus dem Aussenhandel erfolgt durch den Vergleich der Konsum-Wachstums-Grenze im Zustand der Autarkie und in demjenigen des Aussenhandels. Bei Ländern mit derselben einzigen Technik wird ein Land immer vom Handel profitieren, das andere verlieren; bei Ländern mit unterschiedlichen Techniken werden beide verlieren. Wenn jedes Land aus einer gegebenen Anzahl von Techniken wählen kann, werden beide noch immer verlieren, selbst wenn die jeweilige Anzahl die gleiche ist. Diese Tatbestände können sowohl in zentralisierten als auch dezentralisierten Planwirtschaften dadurch vermieden werden, dass bei einer gegebenen Wachstumsrate die Konsummaximierung zum Ziel des Aussenhandels erklärt wird.

RÉSUMÉ

Des techniques productives correspondentes (appliquées avant l'introduction du commerce extérieur) n'empêchent pas des relations de commerce entre les pays, pourvu que leurs taux de profit diffèrent: ceci implique des différences dans les prix relatifs. L'analyse compare les limites salaire-profit appartenantes aux activités productives des équilibres autarciques et à celles du commerce extérieur. On peut dire que le motif du commerce est la poursuite d'une combinaison toujours plus élevée profit-salaire réel. L'estimation des «profits du commerce extérieur» est faite en comparant les limites consomption-croissance et dans l'autarquie et dans le commerce extérieur. Si deux pays ont une même, seule technique, l'un d'entre eux profitera toujours du commerce, tandis que l'autre subira des pertes. S'ils ont des techniques différentes, tous deux pourraient subir des pertes. Lorsque chacun des pays a le choix entre un nombre donné de techniques: pertes possibles pour les deux, même si le nombre choisi est identique. Une économie planifiée – centralisée ou décentralisée – permettrait d'éviter ces pertes en faisant de la maximisation de la consomption (à un taux de croissance donné) l'objectif du commerce extérieur.

[15]

SCARCE NATURAL RESOURCES AND INCOME DISTRIBUTION (*)

by Guido Montani

Summary: Introduction. SECTION I: DIFFERENTIAL RENT – 1. The production system with lands of different qualities; 2. Relations between the rates of profit, wage and rent; 3. Order of fertility and income distribution. SECTION II: RENT ON LAND OF A SINGLE QUALITY – 1. The production system with land of the same quality; 2. Relations between the rates of profit, wage and rent; 3. Possibility of an upward-sloping wage-profit frontier

INTRODUCTION

In traditional economic theory it is customary to define the « scarcity of a factor of production » in relation to the law of decreasing returns. It is said, actually, that « Decreasing returns arise from the *scarcity* of some factor of production and the consequent necessity of using greater and greater proportions of the others along with it » (1). Therefore, the crucial element of this definition of scarcity is the proportion between the factors of production. Given a certain technical knowledge, a factor becomes scarce when, adding up further units of a variable factor to a fixed quantity of the first factor, we obtain decreasing increments of product.

In this paper, on the contrary, the concept of « scarcity » will be defined independently of the proportion between the employed factors. Moreover, the same term of « factor of production » will be dropped so as not to create confusion between the two meanings. In its place we shall speak of « scarce natural resources ».

Here, we shall try to prove that, given certain methods of

(*) I wish to thank prof. P. Garegnani, of the University of Rome, prof. I. Steedman, of the University of Manchester and prof. A. Sdralevich, of the University of Pavia, for comments and criticisms on a previous draft of this paper. I would also thank prof. U. Magnani, of the University of Pavia, who gave me valuable advice over certain mathematical difficulties.

(1) J. M. CASSEL, *On the law of variable proportions*, « Explorations in Economics », 1936, pp. 223-236; reprinted in *The theory of Income Distribution*, Allen and Unwin, 1967, pp. 103-118. Loc. cit. p. 104. The italics are mine.

production, some natural resources, that is to say the means of production employed in the production of other commodities but not themselves produced, become « scarce » according to the produced quantity of a given commodity and to the income distribution between profits and wages. This second approach to the scarcity problem was typical of classical economists, who spoke of scarcity of resources in relation to the needs of the whole economy, not in relation to the quantities of other factors of production (²), and it was resumed by P. Sraffa, some years ago, in his modern formulation of the classical economic theory of value and distribution (³).

The income gained on scarce natural resources is called rent. The Ricardian theory of rent, under the assumption of a given wage rate, has already been discussed (⁴). We shall now return to that analysis on the assumption that income distribution between wages and profit changes, and we shall study the effects of these changes on the level of rent rates and on the « scarcity » of land, the only natural resource in short supply taken into consideration here.

The argument will be divided into two sections. In the first we shall discuss the effects of changes in the distribution between wages and profits in the case of differential rent, i.e. rent on lands of different qualities. In the second section we shall study the same effects for rent on land of a single quality. It is to be understood that the assumption of constant returns to scale is expressly introduced whenever changes in the volume of production are taken into consideration.

The results will confirm some very concise statements made by Sraffa. For instance, for the case of differential rent Sraffa says that the order of fertility of lands « as well as the magnitude of the rents themselves, may vary with the variation of r and w » (⁵), i.e. the order of fertility and rents changes with a change in the distribution of income between profits and wages. And, for the case of rent on land of a single quality, the discussion will prove that the number of methods in use and land « scarcity » depend on income distribution between profits and wages: « if there were no scarcity, only one method, the cheapest, would be used on the land and there could be no rent » (⁶).

(²) A. Marshall, for instance, clearly points out this diversity of points of view: « ... the diminishing return which arises from an ill-proportioned application of the various agents of production into a particular task has little in common with that broad tendency to the pressure of a crowded and growing population on the means of subsistence. The great classical Law of Diminishing Return has its chief application, not to any one particular crop, but to all the chief food crops ». Cf. A. Marshall, *Principles of Economics*, 8th edition, Macmillan, 1972, p. 338.

(³) P. Sraffa, *Production of Commodities by means of Commodities*, Cambridge University Press, 1960. Cf. especially the *Preface*.

(⁴) Cf. G. Montani, *La teoria ricardiana della rendita*, « L'industria », 1972, pp. 221-243.

(⁵) Cf. P. Sraffa, *op. cit.*, p. 75, § 86.

(⁶) Cf. P. Sraffa, *op. cit.*, p. 76, § 88.

— 70 —

1 - Differential rent

1. *The production system with lands of different qualities*

We shall now examine a very simplified economic system. It consists of two sectors: industry and agriculture. Only two commodities are produced, one by « industry » and one by « agriculture ». Industrial production will take place only with a given method of production. In the agricultural sector we assume that the agricultural produce, here labelled simply as « corn », is grown on two different kinds of land (of different quality). Thus we have two methods for corn production used either simultaneously or alternately, according to circumstances.

We assume moreover that wages are paid at the end of the production cycle and production requires only circulating capital. Gross production of each commodity is sufficient to replace the means of production worn out in the course of productive process, and to provide a physical surplus: the system is in a self-replacing state. Net production is distributed between wages, profits and, if that is the case, rents.

Therefore the system of the production equations can be written as follows:

[1] $$A_a p_a (1 + r) + L_a w = A p_a$$

[2] $$A^1_z p_a (1 + r) + L^1_z w + \Lambda^1 \rho^1 = Z^1$$

[3] $$A^2_z p_a (1 + r) + L^2_z w + \Lambda^2 \rho^2 = Z^2$$

The produced quantity of the industrial commodity « a » is A and the quantities of « a » required for the production of A , Z^1 and Z^2, respectively, are called A_a , A^1_z and A^2_z . Corn (Z) can be grown either on land Λ^1 or on land Λ^2. It enters neither into self-production nor into commodity « a » production: therefore the only basic commodity [1] is the industrial one. Labour (L_a , L^1_z , L^2_z) enters directly into the production of every commodity. The rate of wage is w , the rate of profit r and the two rent rates (because the two lands are of different qualities) are ρ^1 and ρ^2 for land Λ^1 and Λ^2 respectively.

Corn has been taken as the standard of value for relative prices and the wage rate. The system consists of three equations and five unknowns: p_a , r , w , ρ^1 and ρ^2. To solve this system it must be remembered that at least one of the two kinds of land is to be taken as redundant in respect of the needs of production. In such a case its rent rate is zero, because no rent can be asked by landlords in the case where availability of land exceeds the agricultural entrepreneurs demand for it. In the case in which ρ^h ($h = 1$ or 2) is zero, the cor-

---

(1) For the definition of basic commodity see P. Sraffa, *op. cit.*, § 6, pp. 7-8.

— 71 —

responding equation for corn production, together with the equation for industrial production, is sufficient to determine the price p_a and the rate of wage w, if *the rate of profit is given exogeneously*. Once these values are known, the rate of rent in the remaining equation can easily be calculated.

Given a certain rate of profit, it is also possible to define an order of fertility for the known kinds of land, that is to say the order in which the different kinds of land enter into cultivation, where it is the case, due to an increased demand for corn. We need only place the equation for industrial production and one of the equations for corn production side by side, on the assumption that the land considered is marginal, and then calculate the value of p_a and w. In our case the experiment can be repeated twice. That land, giving the lowest price of corn — i.e. the highest p_a — or the highest rate of wage, is obviously the first to undergo cultivation by agricultural entrepreneurs and only if corn production must further increase will they raise production on the second type of land too. The land first cultivated is therefore defined as the most fruitful and the order of fertility, in general, will demonstrate the order followed by entrepreneurs to increase cultivation on different lands (2).

2. *Relations between the rates of profit, wage and rent*

We shall study the effects caused by a change in the rate of profit on the assumption that the quantity of corn to be produced is an independent variable; i.e. we shall not take into consideration the causes affecting the volume of production. In this way, given the methods of production, the quantity of land to be cultivated is determined. There are good reasons to believe that this assumption is fundamental for a suitable treatment of classical political economy (1).

We assume that the quantity of corn to be produced is such that both available areas must be cultivated, but that, whatever the case, a certain surface of the total available land is always free and untilled: thus we always have on the one hand a scarcity of land and on the other redundant land. In order to recognize the marginal land, i.e.

---

(2) The order of fertility can be defined in relation either to a given rate of profit or a given wage rate. The order is defined unambiguously only when the indipendent variable (the profit or the wage rate) is chosen, but it can differ according to the chosen variable.

(1) This approach is only apparently strange for the case under discussion in which, owing to a change in the rate of profit, there will also be a change in the price of the two commodities. Therefore the assumption that the quantity demanded of a certain commodity does not change when its price changes would seem an extreme abstraction. This assumption, however, does, as a first step, allow an appropriate study of the relations between the rate of profit, the rate of wage, and prices of scarce natural resources, without undermining the study of the causes affecting the volume of production.

the redundant land, for the use of which no rent is paid, and what is to be the most fruitful land, the scarce land, for the use of which a positive rate of rent must be paid, we can obtain from the previous equations the wage-profit relation assuming that, alternately, one of the two lands is the marginal one. We can write:

$$[4] \quad w_h(r) = \frac{Z^h\,[A - A_a(1+r)]}{L_z^h A + (1+r)\,(A_z^h L_a - L_z^h A_a)} \qquad (h = 1 \text{ or } 2)$$

Since we have two kinds of land available, we also have two relations between the wage rate and the rate of profit. The behaviour and characteristics [2] of one of these relations may be represented as follows:

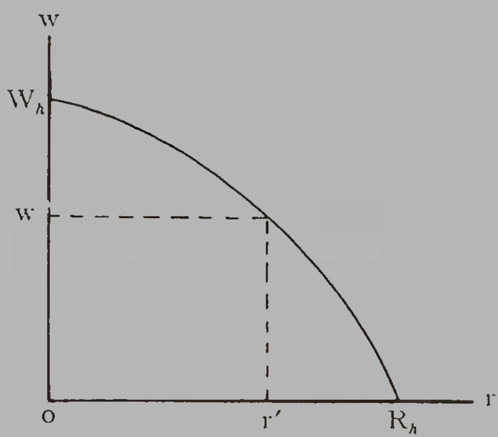


Fig. 1.

When $r = 0$ the whole net product goes to workers. The height OW_h, in fig. 1, shows therefore the value (measured in units of corn) of net product per worker, when the rate of profit in zero. In relation to a zero wage rate we can obtain the maximum rate of profit (R_h) for that economic system. Points W_h and R_h may be joined by a straight, concave or convex line, depending on the proportion between labour and the quantities of commodity « *a* » which enter the production of the two industries [3]. Relation $w_h(r)$ gives the value of the wage rate, given the rate or profit.

Let us obtain now the relation between the rate of rent and the rate of profit. Placing $\rho^1 = 0$ we get:

$$[5] \quad \rho^2(r) = \frac{Z^2\,[L_z^1 A + (1+r)\,(L_a A_z^1 - L_z^1 A_a)] - Z^1\,[L_z^2 A + (1+r)\,(L_a A_z^2 - L_z^2 A_a)]}{L_z^1 A + (1+r)\,(L_a A_z^1 - L_z^1 A_a)}$$

---

(2) For other properties of this system of production see P. Garegnani, *Heterogeneous Capital, the Production Function and the Theory of Distribution*, « The Review of Economic Studies », 1970, pp. 407-436, section I.

(3) Cf. P. Garegnani, *op. cit.*, p. 410.

— 73 —

In a similar way, placing $\rho^2 = 0$ we can easily get function $\rho^1(r)$, which is exactly similar to [5], except for indices 1 and 2 (which show the coefficients of production for lands $\wedge^1$ and $\wedge^2$) which are exchanged.

The behaviour of these rent rates can easily be studied in relation to the behaviour of function $w_1(r)$ and $w_2(r)$. Let us suppose that the two last functions behave as shown in fig. 2a.

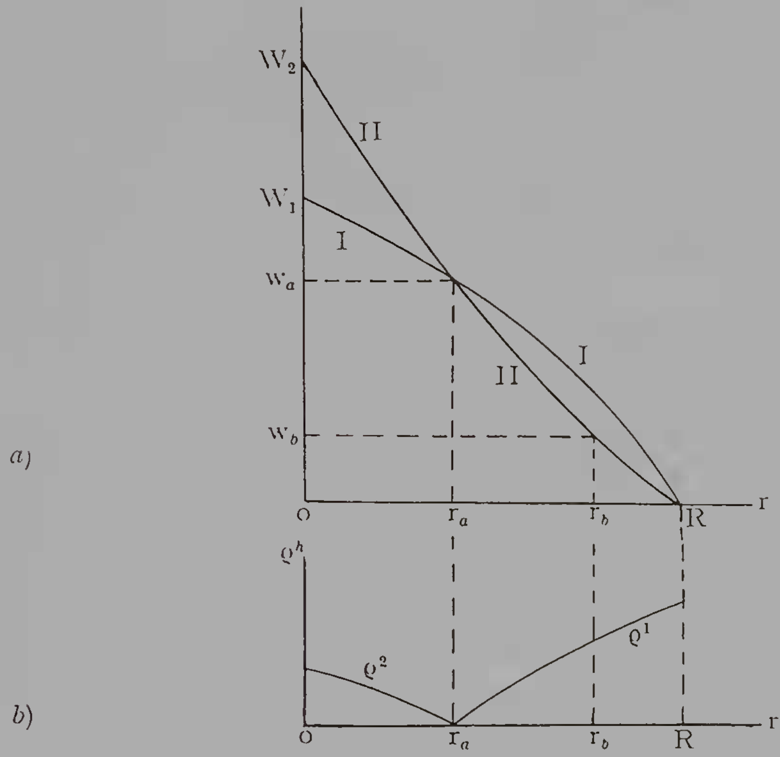


Fig. 2.

In fig. 2a, the behaviour of relation $w_2(r)$ and the behaviour of $w_1(r)$ are shown by II and by I. (One must notice that the two economic systems, I and II, have a common maximum rate of profit. This peculiarity derives from our assumption that in our economy there is only one basic commodity, commodity « *a* » — the method of production of commodity « *a* » is common to both system — thus the maximum rate of profit $\left(R = \dfrac{A - A_a}{A_a}\right)$ is equal to the maximum rate of reproduction for commodity « *a* »).

Let us begin to consider the case in which $r = 0$. Since we assumed that both areas must enter into cultivation, the maximum rate of wage is OW^1, because for higher rates of wage agricultural entrepreneurs are forced to leave land $\wedge^1$ untilled. Nevertheless, on land $\wedge^2$, with this rate of wage OW_1, it is possible to get a positive rate of profit In this case, land $\wedge^2$ is more fruitful than land $\wedge^1$. Competition

— 74 —

among agricultural entrepreneurs to buy up this more fruitful type of land enable landlords of land Λ^2 to ask for rent. The height and positive sign of the rent rate ρ^2 can be ascertained in relation [5]. One can easily see that the sign of the difference between relation II and I, that is to say:

$$\Delta w = w_2(r) - w_1(r)$$

shows also the positivity or negativity of the rent rate ρ^2 (4). Since Δw is, in such a case, positive, also ρ^2, as one can see in fig. 2b, is positive.

When the rate of profit increases from zero to r_a, the rate of wage decreases from OW_1 to w_a. The difference Δw will decrease progressively and the same happens for the rate of rent (ρ^2) on the second land (5).

Profitability in cultivating the first land, compared to the second, diminishes steadily. At the point where the rate of profit is equal to r_a and the rate of wage is w_a, cultivation on land Λ^1 and Λ^2 is equally convenient. Consequently, the total available land for the whole economy will be redundant and no rent can be asked by landlords. For this rate of profit we have $\rho^1 = 0$ and $\rho^2 = 0$ (we could also verify that when $\Delta w = 0$, then $\rho^h = 0$).

For rates of profit falling between r_a and R cultivation on land Λ^1 emerges as more convenient than cultivation on land Λ^2. For instance, at the rate of profit r_b it should be possible to pay wages at higher rates than w_b, should only land Λ^1 be cultivated. But since, by assumption, both lands must enter into cultivation, wages cannot rise above w_b. At this wage rate, nevertheless, agricultural entrepreneurs exploiting land Λ^1 are able to obtain a greater rate of profit than entrepreneurs exploiting only land Λ^2. Competition among agricultural entrepreneurs will therefore place landlords of land Λ^1 in a position to ask for rent. The height of this rent rate, ρ^1, derives from the difference in fertility between the two lands, and its behaviour

(4) If we put $y_1 = L_z^1 A + (1+r)(L_a A_z^1 - L_z^1 A_a)$ and $y_2 = L_z^2 A + (1+r)(L_a A_z^2 - L_z^2 A_a)$, we can write the difference Δw in the following way:

$$\Delta w = \frac{[A - A_a(1+r)]\{Z^2 y_1 - Z^1 y_2\}}{y_1 \cdot y_2}$$

Since $A - A_a(1+r) > 0$, for r included between 0 and R, and since the denominator is always positive for values of r included between 0 and R, the sign of Δw depends only on the sign of { }, which is equal to the numerator of equation [5].

(5) The first derivative of function [5] gives either values which are always positive if $(L_z^1 A_z^2 - L_z^2 A_z^1) < 0$ or always negative if $(L_z^1 A_z^2 - L_z^2 A_z^1) > 0$.

is shown in fig. 2b (we could, moreover, ascertain that when Δw is negative ρ^1 is positive).

When $r = R$ the value of ρ^1 is maximum (6).

3. *Order of fertility and income distribution*

Let us now deal with changes in the order of fertility when the rate of profit changes (and as a consequence the rate of wage too) in a slightly more complicated case (1) than the previous one; i.e. corn is considered as a basic commodity.

Agricultural produce (Z) can be cultivated on three different kinds of land (Λ^1, Λ^2 and Λ^3). Commodity « a » enters into the production of commodity « z » and commodity « z » enters into the production of « a ». If we take corn once more as a unit to measure values, the following system of four equations can be written:

$$[6] \quad (A_a p_a + Z_a)(1 + r) + L_a w = A p_a$$

$$[7] \quad (A_z^h p_a + Z_z^h)(1 + r) + L_z^h w + \Lambda^h \rho^h = Z^h \qquad (h = 1, 2, 3)$$

In this case too, the system provides a physical surplus, whatever the marginal land is. If we successively now place for each of the three equations for corn production ρ^h equal zero, we are able to build three systems (any one system therefore includes the equation for industrial production and one of the equations for agricultural production). Given the rate of profit, each system enables us to determine the wage rate. Since we have at our disposal three lands we shall get three different relations, which we may write in the following way:

$$[8] \quad w_h(r) = \frac{AZ^h - (1 + r)(Z^h A_a + A Z_z^h) - (1 + r)^2 (A_a Z_z^h - A_z^h Z_a)}{L_z^h A + (1 + r)(L_a A_z^h - L_z^h A_a)} \qquad (h = 1, 2, 3)$$

(6) The system of equations in such a case is:

$$[1] \quad A_a p_a (1 + R) = A p_a$$

$$[2] \quad A_z^1 p_a (1 + R) + \Lambda^1 \rho^1 = Z^1$$

$$[3] \quad A_z^2 p_a (1 + R) = Z^2$$

Note that equation [1] determines R for every value of $p_a \neq 0$. Once R is known, equation [3] determines p_a and when R and p_a are known equation [2] will determine ρ^1.

It must be pointed out that at the point in which $r = R$, the value $A - A_a(1 + r)$ at the numerator of Δw is zero. Therefore a zero value of Δw is consistent with a negative value of { } and, consequently, with a positive value of ρ^1.

(1) A case similar to the one under discussion here has already been analyzed by A. Quadrio Curzio, *Rendita e distribuzione in un modello economico plurisettoriale*, Giuffre, Milano, 1967, pp. 196-203.

This relation [2] can be seen (in fig. 3) in a diagram similar to fig. 1. When the rate of profit is zero, all net production goes to workers (OW_h will be therefore the maximum rate of wage). For a zero wage rate, we can get the maximum rate of profit (R_h). Functions $w_h(r)$ can be represented by a straight, concave or convex curve [3].

Let us now show simultaneously relations $w_h(r)$ for the three systems in the same diagram.

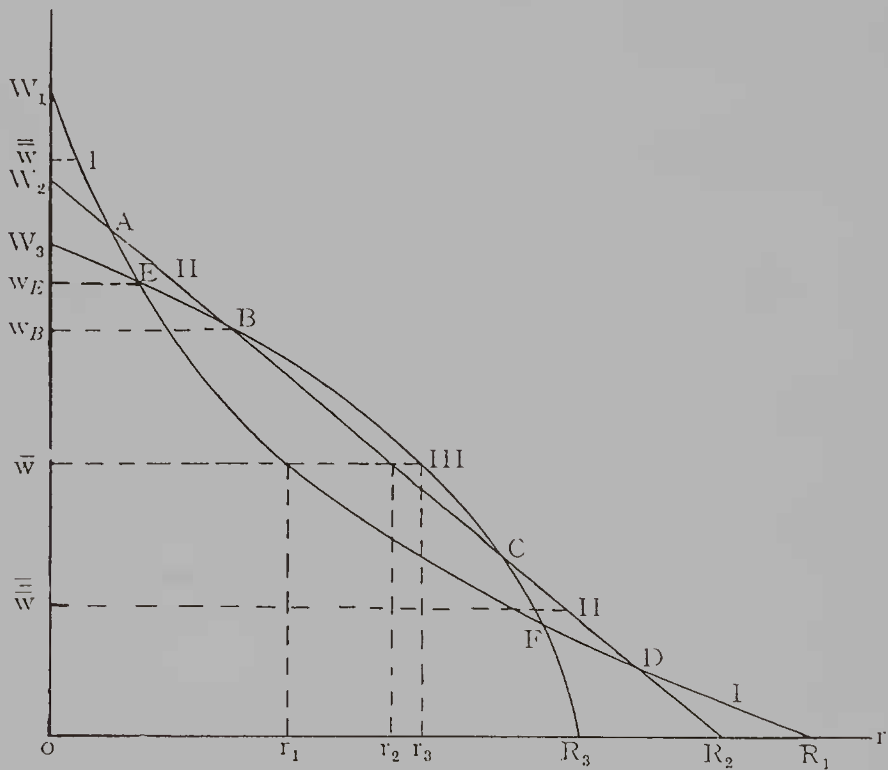


Fig. 3.

The problem, now, is how to choose a certain method for corn production, for a *given wage rate* (an exception is made to the rule that the rate of profit is the independent variable purely on grounds of expediency: the explanation of the working of the economic system is simplified and the mathematical relations $w_h(r)$ are, on the other

(2) Relation $w_h(r)$ has been studied from an economic point of view by J. Robinson and K. A. Naqvi, *The Badly Behaved Production Function*, « The Quarterly Journal of Economics », 1967, pp. 579-591, principally pp. 585-588. For a numerical example, concerning a similar case, but with advanced wages and unit coefficients, see the *Appendix* of P. Garegnani. *Switching of Techniques*, « The Quarterly Journal of Economics », 1966, pp. 554-567.

(3) For conditions of linearity, convexity and concavity see J. Robinson and A. K. A. Naqvi, *op. cit.*

hand, always the same). What concern us is to choose which one of the agricultural equations enters into the system of equations, side by side with the equation for industrial production. The equation chosen will be the one relating to marginal land, the land without rent.

The three relations $w_h(r)$, shown in fig. 3, are not enough to determine the marginal land. We also need to know the quantity of corn to be produced in order to say whether only one area is to be raised or whether two or three areas must be put into cultivation.

Let us now make the assumption that the quantity of corn to be produced is a very small quantity, so that every type of land is in a position to satisfy the needs of society. In order to know which one of the three types of land is being cultivated, let us take a wage rate equal to $\bar{w}$. From fig. 3 we can see that, according to whether cultivation is raised on the first, second or third kind of land, the possible rates of profit are $r_1 < r_2 < r_3$. There is no doubt that, at wage rate $\bar{w}$, only the third area is cultivated. If the wage rate were to go up to $\bar{\bar{w}}$, on the contrary, only the first area could be cultivated (at wage rate $\bar{\bar{w}}$, production on land Λ^2 and Λ^3 does not consent to any positive rate of profit).

The increase in the wage rate from $\bar{w}$ to $\bar{\bar{w}}$, has caused a change in the order of fertility of the three types of land. Between point B and point A, in fact, the second type of land emerges as the most fruitful, while above point A cultivation is profitable only on the first type of land. The same changes (and in the same order) would happen for a lowering of the wage rate from $\bar{w}$ to zero. Relations $w_h(r)$ do not show only the most fruitful land, but also the whole order of fertility, for a given wage rate. At wage $\bar{w}$ the order of fertility gives land Λ^3 as the first (the most fruitful one), land Λ^2 as the second and land Λ^1 as the third fruitful land. At a wage rate equal to $\bar{\bar{\bar{w}}}$, land Λ^2 is the most fruitful, land Λ^3 is the second and the last fruitful land is Λ^1. We can therefore assert that the order of fertility is defineble only for a given rate of wage (or profit) and that when the rate of wage (or profit) changes, the order of fertility changes too. Therefore it is not possible to measure fertility in physical terms, because it does not depend merely on « generosity of nature ».

Let us consider now the contrary case, in which the quantity of corn to be produced is such that all the three areas must be cultivated but one is not cultivated completely. We always begin with a wage rate equal to $\bar{w}$. At this rate, land Λ^1 is the marginal one, i.e. the land giving the lowest rate of profit and the equation of this land together with the industrial production equation forms the system which determines relation $w_1(r)$. On this marginal land the rate of rent is zero. On land Λ^2 and Λ^3 it is not possibile to obtain a higher rate of profit, such as one might deduce from fig. 3, but, instead, a positive rent, because competition among agricultural entrepreneurs will place landlords in a position to ask for a positive rent.

— 78 —

The rate of rent payable to landowners of land Λ^2 and Λ^3 is easily calculable. From equation [7] we get:

$$[9] \qquad \rho^h = \frac{A_z^h p_a + Z_z^h}{\Lambda^h}\left[\left(\frac{Z^h - L_z^h w}{A_z^h p_a + Z_z^h} - 1\right) - r_1\right] \qquad (h = 2,\ 3)$$

These two rates of rent are certainly positive (4). Nevertheless we cannot know, on the basis of equation [9] alone, and without knowing the exact value of the coefficients, whether the rate of rent belonging to land Λ^2 is greater than the rate belonging to land Λ^3. It is possible to show (5) that price p_a lowers when cultivation is extended from the third to the first least fruitful land. The lowering of price p_a, if coupled with suitable values of technical coefficients, may cause a reversal of order of fertility and the order of rentability (in our case it could happen that $\rho^2 > \rho^3$ even if $r_3 > r_2$).

For similar reasons, if we consider a wage rate w_B, we do not obtain an equal rate of rent on lands Λ^2 and Λ^3, as it might seem at first glance. In point B, agricultural entrepreneurs who might, possibly, cultivate land Λ^2 and Λ^3 without paying rent, would have the same rate of profit. But they would be forced by competition, sooner or later, to pay a rent that would be higher the greater the difference between productivity per unit of land and cost per unit of land (6). If we concede that productivity per unit of land is the same for both areas, when the proportion between labour and means of production is different for the two methods, the lowering of the rate

---

(4) If land Λ^3 were the marginal land (as in the previous case in which a small quantity of corn is produced) the rate of profit would be:

$$\text{I)} \qquad r_3 = \frac{Z^3 - L_z^3 w}{A_z^3 p_a^{(3)} + Z_z^3} - 1$$

where $p_a^{(3)}$ is the price of commodity « a » when Λ^3 is the marginal land. If cultivation is extended to the first land, on land Λ^3 a rent of the following value will appear:

$$\text{II)} \qquad \rho^3 = \frac{A_z^3 p_a^{(1)} + Z_z^3}{\Lambda^3}\left[\left(\frac{Z^3 - L_z^3 w}{A_z^3 p_a^{(1)} + Z_z^3} - 1\right) - r_1\right]$$

We can certainly say that $\rho^3 > 0$. Indeed, the round bracket in II is simply the rate of profit in I when the rate of rent is zero: the only difference is that in I there is $p_a^{(3)}$ instead of $p_a^{(1)}$. But it can be shown (see footnote 5) that $p_a^{(1)}$ is greater than $p_a^{(3)}$. If we substitute $p_a^{(3)}$ in I with $p_a^{(1)}$ we get a rate of profit greater than r_3. Since the sign of ρ^3 depends only on the sign of the square bracket in II, knowing that $r_3 > r_1$, we can conclude that $\rho^3 > 0$.

This proof is taken from A. Quadrio Curzio, *op. cit.*, p. 85 footnote.

(5) For a general proof see G. Montani, *op. cit.*, Section I, § 3

(6) The value of the rent rate is $\rho^h = \frac{Z^h}{\Lambda^h} - \frac{C^h}{\Lambda^h}$ where $C^h = L_z^h w + (1 + r)(A_z^h p_a + Z_z^h)$.

of profit to the level carried out on the marginal land causes two different rent rates.

When the wage rate goes up to w_E we can see that on the third land any possibility of obtaining rent disappears. Fertility of this land is, so to say, reduced to the level of the first land's fertility. At a wage rate greater than w_E rent will appear on the first land and the third land will become the marginal one.

Cultivation on the three lands may continue up to the point where the rate does not exceed W_3. For wages greater than W_3, production on the third land will not be profitable and will be given up. The quantity of corn available for the economy will be reduced to that raised on lands Λ^1 and Λ^2. It seems right to say therefore that, when all three lands are to be cultivated, W_3 represents the maximum wage rate for the economy. Similarly, we can repeat the same argument for the rate of profit. When it is necessary to raise cultivation on the three lands, R_3 represents the maximum rate of profit for the economy (7).

Therefore, the relation between the rate of profit and the rate of wage is given by W_3EFR_3, when all three lands are cultivated. This relation defines the maximum rate of profit (or wage) for the economy once a certain rate of wage (or profit) is established at a certain level. The relation between wages and profits may, nevertheless, change according to the scale of cultivation. If it is possible to cultivate only one land, then obviously the cultivated land will be the most fruitful. In such a case, relation $w(r)$ is defined by the external side of the three relations $w_h(r)$. It will be W_1ABCDR_1. This could also be called the « frontier of fertility », because it defines, for a given rate of wage (or profit), the most fruitful land.

From the foregoing remarks we can draw the conclusion that Ricardo's statement that « Rent is the portion of the produce of the earth, which is paid to the landlord for the use of the original and indestructible powers of the soil » (8) is not completely true. The « original and indestructible powers of the soil », that is, the coefficients of production in our terminology, comprise only one of the conditions which affect the rent of the soil; the other condition, given the total quantity of corn to be produced, is income distribution between profits and wages. The same consideration must be made for « scarcity » of land. As we have seen, a certain land may be either scarce or redundant in relation to a certain quantity of the goods produced for the market and according to the diverse values of the rate of wage and profit.

---

(7) As it is shown by equation [8], for $w = 0$ one no longer obtains an equal maximum rate of profit for the three systems. Here corn enters in commodity « *a* » production, which is therefore no longer the only basic commodity.

(8) D. Ricardo, *On the Principle of Political Economy and Taxation*, in « The Works and Correspondence of David Ricardo », vol. I, Cambridge University Press, 1951, p. 67.

II - Rent on land of a single quality

1. *The production system with land of a single quality*

To facilitate the study of causes which occasion rent on land of the same quality and, consequently, land scarcity, let us first consider a very simplified system for the production of corn and one industrial commodity, like the one utilised in precedence to study the differential rent. Exceptionally, and only for this paragraph, we take the industrial commodity as the standard of value. This device will considerably simplify the discussion.

The production equations are:

$$A_a (1 + r) + L_a w = A \tag{10}$$

$$A_z^1 (1 + r) + L_z^1 w + \Lambda^1 \rho = Z^1 p_z \tag{11}$$

$$A_z^2 (1 + r) + L_z^2 w + \Lambda^2 \rho = Z^2 p_z \tag{12}$$

Obviously we no longer have different lands. *All the land is of the same quality* and, consequently, there will be only one rate of rent (ρ). Quantities Λ^1 and Λ^2, therefore, no longer symbolize quantities of different lands, but different quantities of the same land. The system includes three equations and four unknowns: r, w, ρ and p_z. Considering the rate of profit (or wage) as given, we are able to determine the value of the three unknowns. The system produces a physical surplus of commodities « *a* » and « *z* ».

We assume that the known methods for corn production are only two and that physical land productivity of the first method is lower than the second method productivity $\left(\frac{Z^2}{\Lambda^2} > \frac{Z^1}{\Lambda^1}\right)$. Moreover, total demand for corn may be satisfied either by raising produce on the whole land, partly by the first method and partly by the second, or by cultivating only a portion of the land available with the second method, leaving the other portion of land untilled (therefore total demand for corn cannot be satisfied by cultivating the available land with the first method only).

Since we have chosen « *a* » as a standard of value for wages, equation [10] is sufficient to determine the wage-profit relation. This relation, which will be linear, may be written down as follows:

$$w(r) = \frac{A - A_a (1 + r)}{L_a} \tag{13}$$

Given the value of the wage rate and the rate of profit, the last two equations enable us to determine the price of corn and the rate of rent.

The two methods for corn production must satisfy the economic

— 81 —

condition of giving no negative rent. This implies that the method with a higher physical land productivity $\left(\frac{Z^h}{\Lambda^h}\right)$ must also have costs per unit of product higher than the less productive method (¹) — where « costs » include the sum total of wages, the value of the means of production and profits (calculated at the natural rate of profit, as it results from equation [10]). We can see graphically this condition in fig. 4, where the relation between the rate of rent and the price of corn is drawn:

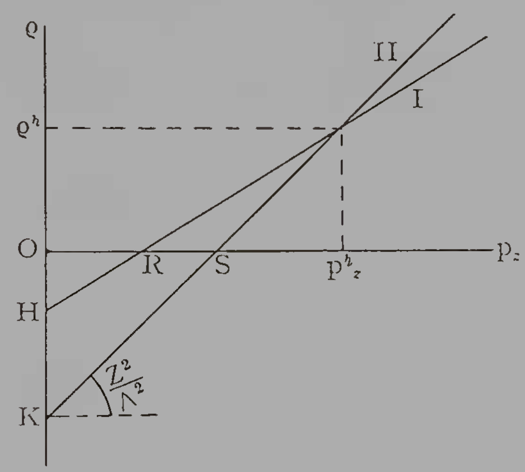


Fig. 4.

The price-rent relation for equation [11] is called I and the price-rent relation for equation [12] is called II. The gradient of the two lines is the physical land productivity for the first and the second method for corn production and the distances OR and OS are the costs per unit of product for method I and II respectively (²).

Let us now suppose, for a moment, that with a given rate of wage and a given rate of profit (determined in equation [10]) only method II is known. Total demand for corn may be satisfied cultivating only a portion of the disposable land with method II and leaving the other portion untilled. The price of corn will be equal to OS and there will be no positive rent. Imagine now that method I is discovered. Method I has a lower physical productivity but it has also lower costs per unit of product, so, at the ruling price OS, some agricultural

(¹) See SRAFFA, *op. cit.*, § 87.

(²) If we put $C^1 = L^1_z w + A^1_z (1 + r)$, then the price-rent relation is $\rho = p_z \frac{Z^1}{\Lambda^1} - \frac{C^1}{\Lambda^1}$. Since OH is equal to $\frac{C^1}{\Lambda^1}$, then $OR = \frac{C^1}{\Lambda^1} \cdot \frac{\Lambda^1}{Z^1} = \frac{C^1}{Z^1}$; that is, the distance OR is equal to the cost per unit of product.

— 82 —

entrepreneur may find convenient either to substitute method II with method I or to begin cultivation with method I on untilled land: at that price of corn entrepreneurs employing method I will obtain extra profits. Of course, in such a condition, every one will employ method I but, since by assumption total demand of corn cannot be satisfied cultivating the disposable land with method I only, land will become scarce and some entrepreneur, in order to obtain land, will offer to landlords a rent, until he is able to obtain a rate of profit (net of rent) higher than the natural rate of profit. Only at price p^n_z of corn method I and method II may coexist; corn will be raised with the two methods jointly and there will be a uniform rate of rent ρ^n.

Let us now consider the case in which a change in the distribution of income between wages and profits is possible. It is convenient, in such a case, to study the behaviour of costs per unit of product for the two methods of corn production, for changes in the rate of profit (or wage). In our case costs (C^h) per unit of product are:

$$[14] \quad \frac{C^h}{Z^h} = \frac{L^h_z A + (1+r)(L_a A^h_z - L^h_z A_a)}{Z^h L_a} \qquad (h = 1,\ 2)$$

We are now in a position to draw two diagrams, one in relation to the other, indicating the behaviour of w (r) and that of costs per unit of product.

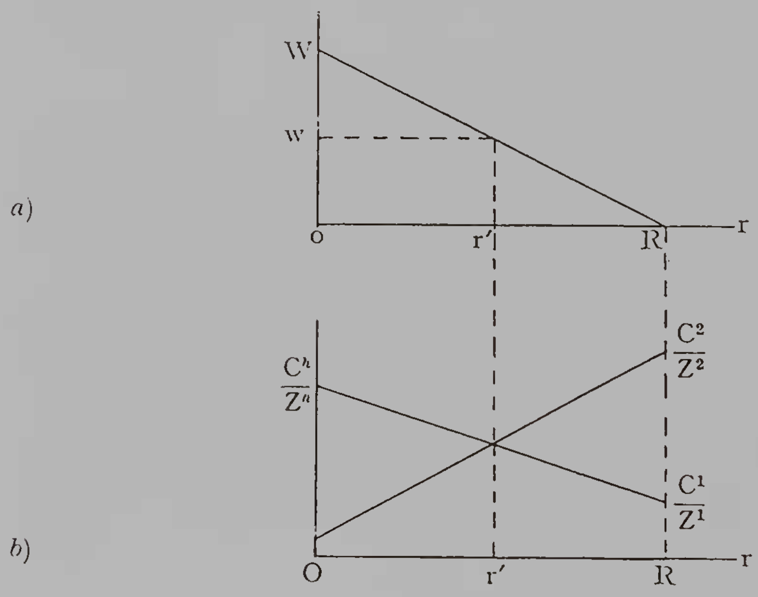


Fig. 5.

We assume that costs per unit of product behave in such a way as to meet at a point somewhere between O and R. The value of the rate of profit which equalizes costs per unit of product for the two

— 83 —

methods has been indicated by r' in fig. 5. In such a case there cannot be any rent and land will be redundant. Since method I and method II are equally convenient, entrepreneurs will adopt method II with a higher productivity (or a combination of methods I and II, but land must be redundant), because by employing method I only land will become scarce and landlords will be able to ask a rent for the use of the soil (in such a case entrepreneurs employing method I will be able to pay a rate of rent equal to the one paid by entrepreneurs employing method II only if they sell corn at a higher price).

For rates of profit included between r' and R costs per unit of product for the second method are higher than costs per unit of product for the first method. Such a case is represented in fig. 4 in which $OS > OR$. There will be a positive rate of rent and corn will be raised on all the disposable land with method I and II jointly used.

For rates of profit falling between O and r' method II, with a higher physical productivity, has also costs per unit of product lower than method I. Nobody will use method I. Only a portion of the disposable land will be cultivated and there will be no rent.

To conclude, when cost per unit of product of the method with a higher productivity is greater than that of the method with lower productivity, competition among entrepreneurs will make the land « scarce » and will enable landlords to ask a rent for the use of the soil. Land scarcity depends therefore on two conditions. The first is that there is at least one method, among those available, for which productivity per acre is such that, if the whole land is cultivated by this method, the quantity of corn to be produced to meet the needs of society cannot be attained. Secondly, it is necessary that this method, the use of which causes a scarcity of land, has a cost per unit of product lower than the unit cost of all methods with a greater productivity. Since this second condition might or might not take place depending on how high the rate of profit (or wage) is, it seems right to conclude that land scarcity depends not only on the extent of the total demand for corn in relation to the methods available, but also on income distribution.

2. *Relations between the rates of profit, wage and rent*

We now abandon our device introduced in the preceding paragraph; the standard of value for wages will once again be corn. Thus the system of production equations may be written as follows:

[15] $$A_a p_a (1 + r) + L_a w = A p_a$$

[16] $$A^1_z p_a (1 + r) + L^1_z w + \Lambda^1 \rho = Z^1$$

[17] $$A^2_z p_a (1 + r) + L^2_z w + \Lambda^2 \rho = Z^2$$

The change in the standard of value is essential. The system of the three production equations can now no longer be broken up into sub-systems. Equation [15] is no longer sufficient to define the wage-profit relation: since the standard of value is corn, relation $w(r)$ is not independent of the equations for corn production.

We retain, however, the assumption of the quantity of corn to be produced: it can be raised either by tilling the whole land partially in accordance with the first method (with a lower productivity than the second) and partially with the second one, or by cultivating a portion of the available land by the second method alone, leaving a fraction of land untilled.

It should be pointed out that from the system of equations three wage-profit relations can be drawn and that it will be necessary to take into consideration the three relations in order to give a satisfactory explanation of the connection between the three rates of remuneration. The first wage-profit relation concerns the case in which production takes place utilizing the two methods jointly. In such a case, by assumption, land will be scarce and there will be a positive rate of rent. When production takes place adopting only the second method on a portion of the available land, on the contrary, there will be no rent: the wage-profit relation is, in a similar case, different from the previous one. Finally, we also need to take into consideration the wage-profit relation in the case in which only the first method is adopted. Such a case is of indirect interest, because the first method will never be able, if adopted alone, to give a sufficient quantity of corn. Nevertheless, this third relation, compared to the second one, will be useful to clarify the mechanism originating land scarcity and rent.

Let us consider the relation between the rate of wage and the rate of profit when the two methods are adopted jointly and there is a positive rate of rent. This relation is:

[18]

$$w_{1,2}(r) = \frac{[Z^1\Lambda^2 - Z^2\Lambda^1]\,[A - A_a(1-r)]}{\Lambda^2\,[L_z^1 A + (1+r)(L_a A_z^1 - L_z^1 A_a)] - \Lambda^1\,[L_z^2 A + (1+r)(L_a A_z^2 - L_z^2 A_a)]}$$

Relation $w_{1,2}(r)$ represents a rectangular hyperbola (1). It cuts

(1) In equation [18] the numerator is negative since $\frac{Z^2}{\Lambda^2} > \frac{Z^1}{\Lambda^1}$. In order to obtain a positive rate of wage we need a negative denominator (which we shall call $y = \Lambda^2 y_1 - \Lambda^1 y_2$; where $y_1 = L_z^1 A + (1+r)(L_a A_z^1 - L_z^1 A_a)$ and similarly for y_2. Finally its derivative is called y').

The first derivative of equation [18] is:

$$\frac{dw}{dr} = \frac{A L_a (Z^1\Lambda^2 - Z^2\Lambda^1)(\Lambda^1 A_z^1 - \Lambda^2 A_z^2)}{y^2}$$

— 85 —

the abscissae axis at point $R = \frac{A - A_a}{A_a}$, the maximum rate of profit, and the ordinate at point $W_{1,2}$ where the wage rate is maximum. The behaviour of relation $w_{1,2}(r)$, in the Nord-East quadrant (²), is in no way dissimilar to the one already drawn in fig. 1. For values of the rate of profit included between O and R the price p_a will always be positive, in relation to a positive wage rate (³). It increases when relation $w_{1,2}(r)$ is concave, it decreases when $w_{1,2}(r)$ is convex.

Before studying the behaviour of the rate of rent we need to take into consideration the two other systems of equations that will give us two other wage-profit relations. We obtain the second system from equation [15] and [17] after having established the rate of rent as equal to zero: that is the case in wich only the second method, more productive than the first, is employed for corn production and land will be redundant. Relation $w_2(r)$ in such a case is:

$$[19] \qquad w_2(r) = \frac{Z^2 A - A_a(1+r)}{L_z^2 A + (1+r)(L_a A_z^2 - L_z^2 A_a)}$$

The behaviour of $w_2(r)$ is represented by a decreasing curve (concave or convex), in the Nord-East sector (⁴), with a maximum wage

---

The sign of this derivative depends only on the expression $(\wedge^1 A_z^1 - \wedge^2 A_z^1)$. Therefore:

$$\frac{dw}{dr} > o \text{ if } \frac{A_z^1}{\wedge^1} > \frac{A_z^2}{\wedge^2} \quad ; \quad \frac{dw}{dr} < o \text{ if } \frac{A_z^1}{\wedge^1} < \frac{A_z^2}{\wedge^2}$$

The second derivative is:

$$\frac{d^2w}{dr^2} = \frac{-2\,AL_a(Z^1\wedge^2 - Z^2\wedge^1)(\wedge^1 A_z - \wedge^2 A_z^1)\,y'}{y^3}$$

(²) The case in which $\frac{dw}{dr} > o$ must be discarded because, for the range of profit rates included between O and R, it implies a negative wage rate and a negative price p_a.

(³) The relation between p_a and r is:

$$p_a = \frac{L_a(Z^1\wedge^2 - Z^2\wedge^1)}{y}$$

The numerator is always negative, but also the denominator is, by assumption, negative, if we require $w > 0$.

(⁴) The first derivative of equation [19] is negative for values of r falling between O and R.

$$\frac{dw}{dr} = \frac{-Z^2 A A_z^2 L_a}{[L_z^2 A + (1+r)(A_z^2 L_a - L_z^2 A_a)]^2} < o$$

— 86 —

equal to W_2, for $r = 0$, and with a positive maximum value of r, equal to $R = \frac{A - A_a}{A_a}$, for $w = 0$.

The third system includes equations [15] and [16] when the rate of rent is zero (formally, relation $w_1(r)$ is completely similar to [19]). This system is never active alone in a position of equilibrium owing to its low productivity per acre.

Finally, we can represent the three relations between the wage rate and the rate of profit in the same diagram, since for each system we have taken corn as the standard of value. At the same time, we can draw the corresponding behaviour of the rate of rent in a second diagram.

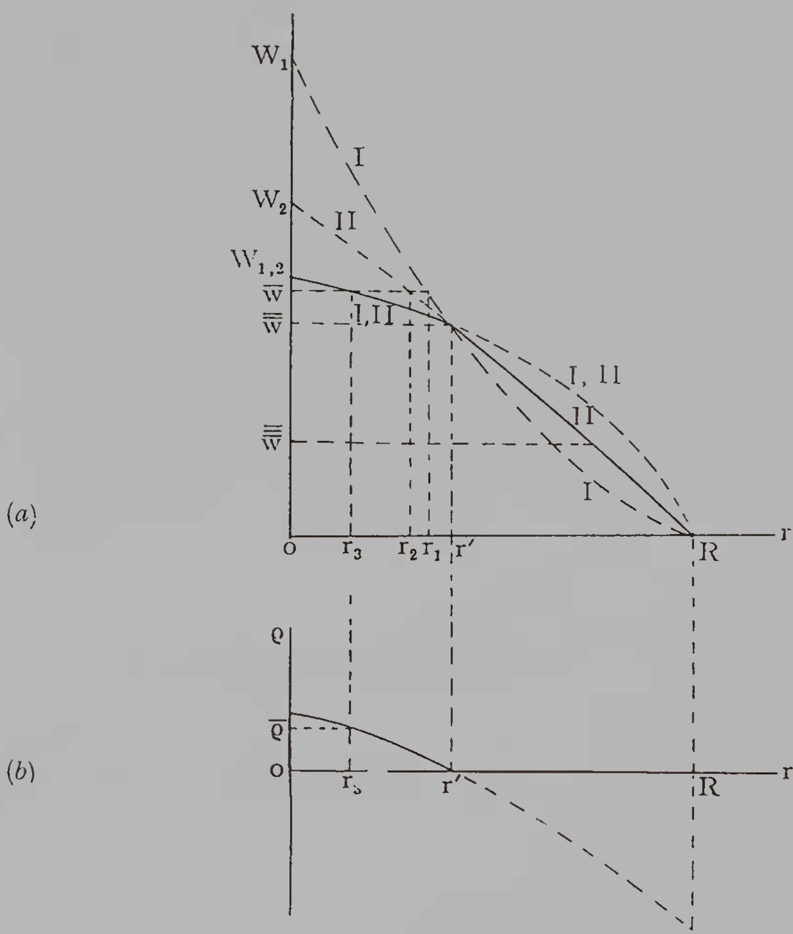


Fig. 6.

The second derivative is:

$$\frac{d^2w}{dr^2} = \frac{(A_z^2 L_a - L_z^2 A_a)\, 2\, Z^2 A A_z^2 L_a}{y^3_2}$$

— 87 —

The behaviour of the rate of rent for changes in the rate of profit, illustrated in fig. 6b, may be drawn of course by the system of the three production equations, when corn produce is obtained by a joint adoption of the two methods. This relations is:

[20]

$$\rho(r) = \frac{Z^2 [L^1_z A + (1+r)(L_a A^1_z - L^1_z A_a)] - Z^1 [L^2_z A + (1+r)(L_a A^2_z - L^2_z A_a)]}{\Lambda^2 [L^1_z A + (1+r)(L_a A^1_z - L^1_z A_a)] - \Lambda^1 [L^2_z A + (1+r)(L_a A^2_z - L^2_z A_a)]}$$

The rate of rent may either increase or decrease when the rate of profit increases. Moreover it can be negative (i.e. the two methods cannot be used jointly) for certain values of r falling between O and R (5).

Let us now examine the economically meaningful values for the rates of profit, wage and rent.

For a profit rate equal to zero we obtain the maximum wage rate for the three systems. As it can be seen from fig. 6a, the maximum rates of wage are $W_1 > W_2 > W_{1,2}$. If we now consider a given wage rate, for instance $\bar{w}$, we learn which system will be in use and what values the rent rate will assume. At the wage rate $\bar{w}$, method I emerges as more convenient than method II $(r_1 > r_2)$. But since method I has too low a productivity to satisfy the demand for corn, land will become scarce and production will have to take place with method I and II jointly. At wage rate $\bar{w}$ agricultural and industrial entrepreneurs will have to be content with a rate of profit r_3, while landlords will obtain a positive rate of rent (6) equal to $\bar{\rho}$. The maximum wage rate is, in such a case, $W_{1,2}$.

At the wage rate $\bar{\bar{w}}$ the systems I and II (and hence I, II) consent the same rate of profit. In such a situation there can be no rent (7). Since the two methods are equally convenient it follows that if, due

---

therefore, recalling that the denominator is always positive for the values of r which have been taken into consideration, the sign of the second derivative depends only on $(A^2_z L_a - L^2_z A_a)$.

(5) The first derivative of [20] is:

$$\frac{d\rho}{dr} = \frac{L_a A\,(Z^1\Lambda^2 - Z^2\Lambda^1)\,(L^2_z A^1_z - L^1_z A^2_z)}{y^2}$$

(6) The sign of the rate of rent may easily be determined by writing equation [20] as follows:

$$\rho(r) = \frac{w_2(r) - w_1(r)}{\Big[A - A_a(1+r)\Big]\left\{\dfrac{\Lambda^2 y_1 - \Lambda^1 y_2}{y_1 \cdot y_2}\right\}}$$

the sign of { } is negative by assumption (see footnote 1). The rate of rent therefore is:

$$\rho > 0 \text{ if } w_1(r) > w_2(r) \quad ; \quad \rho < 0 \text{ if } w_1(r) < w_2(r)$$

(7) The value of r' can be deduced by equalizing the equations [18] and [19]. The same value of r' will emerge by placing $\rho = 0$ in equation [20]. Moreover, the same value can again be found by equalizing the relations $w_2(r)$ and $w_1(r)$.

to some error, land becomes scarce entrepreneurs will at once replace the first with the second method, with higher productivity and making land redundant again.

For a wage rate included between $\bar{\bar{w}}$ and zero, for instance $\bar{\bar{\bar{w}}}$, there can be no positive rent. The second method is now more convenient than the first and entrepreneurs will certainly adopt it. Only a portion of available land will be cultivated and no landlord will be able to ask for rent. For values of r falling between r' and R the rate of rent will therefore be equal to zero and the wage-profit relation will be defined by [19].

In conclusion, we can imagine, to complete the survey, a further case in which the rate of rent increases for certain values of the rate of profit [8]. The behaviour of the three wage-profit relations and the correspondent rate of rent is represented in fig. 7.

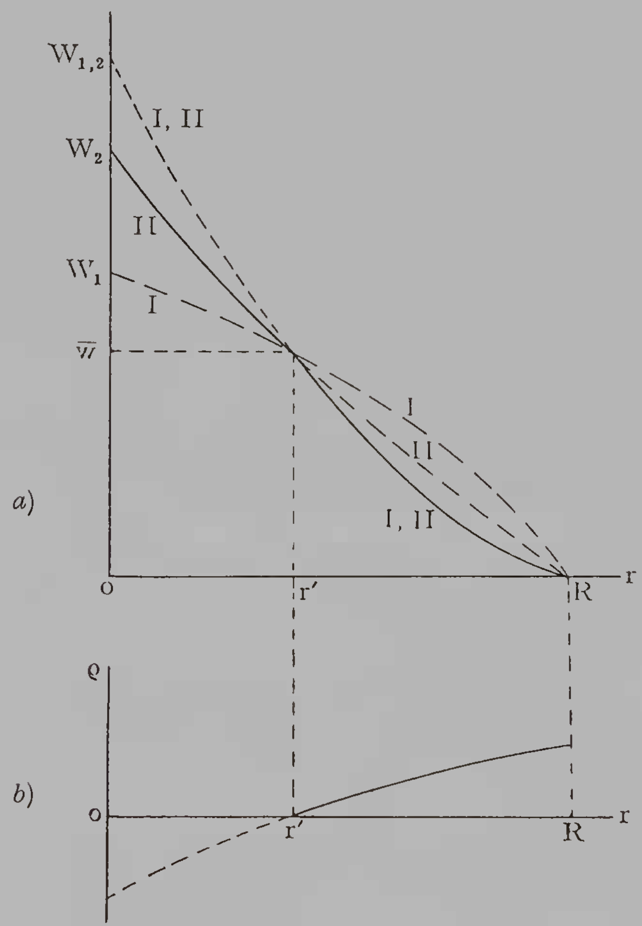


Fig. 7.

(8) A numerical example of a similar case is discussed in the *Appendix* (Example 3).

For wage rates falling between W_2 and $\overline{w}$ only the second method will be active and there will be no rent. For wages included between $\overline{w}$ and zero method I and II will enter jointly into activity and a positive rate of rent will appear.

3. *Possibility of an upward-sloping wage-profit frontier*

We now introduce some slight complications into the production system of the previous paragraph (but leave unchanged the assumption about the produced quantity of corn and productivities of the two methods): we consider corn as a means of production (however, to simplify the problem, we make the additional assumption that corn enters only in commodity « *a* » production). The system, when there is a rent, can be represented as follows:

$$[21] \qquad (A_a p_a + Z_a)(1 + r) + L_a w = A p_a$$

$$[22] \qquad A^1_z p_a (1 + r) + L^1_z w + \wedge^1 \rho = Z^1$$

$$[23] \qquad A^2_z p_a (1 + r) + L^2_z w + \wedge^2 \rho = Z^2$$

Here too, as in the previous case, corn has been taken as the standard of value and the quantities produced of every commodity are greater than quantities employed as means of production. If we take the rate of profit as the independent variable, the system includes three equations and three unknowns $(p_a\,,\ w\,,\ \rho)$. The wage-profit relation is:

[24]

$$w_{1,2}(r) = \frac{A\,(Z^1\wedge^2 - Z^2\wedge^1) - A_a\,(Z^1\wedge^2 - Z^2\wedge^1)\,(1+r) - Z_a\,(\wedge^2 A^1_z - \wedge^1 A^2_z)\,(1+r)^2}{\wedge^2\,[L^1_z A + (1+r)\,(L_a A^1_z - L^1_z A_a)] - \wedge^1\,[L^2_z A + (1+r)\,(L_a A^2_z - L^2_z A_a)]}$$

This relation, compared to relation $w_{1,2}(r)$ of the previous paragraph, presents the novelty that, for certain values of the technical coefficients, an increase in the wage rate is possible together with an increase in the profit rate (but, of course, for different values of the technical coefficients the usual wage-profit relation [1] is still possible). This anomalous wage-profit relation may accur only in the case of production with the two methods jointly: i.e. when there

---

[1] An example in which there is normal behaviour of the wage-profit relation is shown in the numerical *Appendix* (Example 4).

— 90 —

is a rent (²). Its behaviour can be seen in fig. 8 in which $W_{1,2}$ is the wage rate for a zero rate of profit (³).

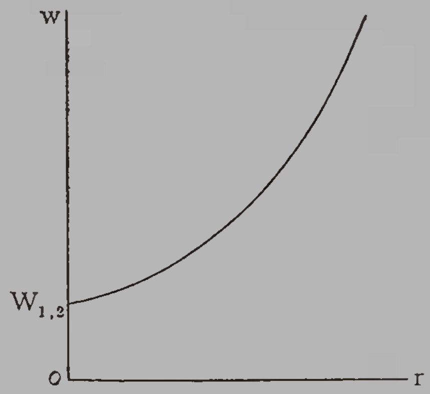


Fig. 8.

We must now explain why he have such an anomaly. Looking at equation [21] we see that, in order to obtain a simultaneous increment of the wage rate and also of the profit rate, a rise in price p_a is inevitable. In equations [22] and [23] it is clear that, in the case in which there is no rent, a simultaneous increment of the three variables is impossible. But if there is a positive rate of rent — and two methods for corn production are in activity — it may happen that when profits and wages increase together, their increment is balanced by a proportionate fall in the rent rate.

Let us now take a look at the other two systems. If the demand of corn may be satisfied cultivating land partly with the first method and partly with the second, it will certainly be possible to produce the same quantity adopting only the second method. In such a case there will be no rent. Equations [21] and [23], for $\rho = 0$, forms a second system with the following wage-profit relation:

$$[25] \qquad w_2(r) = \frac{Z^2[A - A_a(1+r)] - A^2_z Z_a (1+r)^2}{L^2_z A + (1+r)(L_a A^2_z - L^2_z A_a)}$$

(²) Cfr. P. Sraffa, *op. cit.*, § 71-72; pp. 61-62.

(³) For $r = 0$, we obtain from equation [24]:

$$W_{1,2} = \frac{(A - A_a)(Z^1\Lambda^2 - Z^2\Lambda^1) - Z_a(\Lambda^2 A^1_z - \Lambda^1 A^2_z)}{y}$$

The value of $W_{1,2}$ could be either positive or negative. The first derivative of [24] is:

$$\frac{dw}{dr} = \frac{(\Lambda^1 A^2_z - \Lambda^2 A^1_z)\{L_a A(Z^1\Lambda^2 - Z^2\Lambda^1) + 2 Z_a A(\Lambda^2 L^1_z - \Lambda^1 L^2_z)(1+r) + Z_a y'(1+r)^2\}}{y^2}$$

where y and y' have the same meaning as in footnote 1 of the previous paragraph.

This relation for values of r falling between O and R is decreasing ([4]).

Finally, we can consider the third system, the one composed by equations [21] and [22], for $\rho = 0$, even if in a position of equilibrium it could never be adopted owing to its low productivity. Relation $w_1(r)$ for this system is formally analogous to relation [25].

We are now able to represent in the same diagram the three relations between the wage rate and the rate of profit. At the same time, we can draw the relating behaviour of the rate of rent.

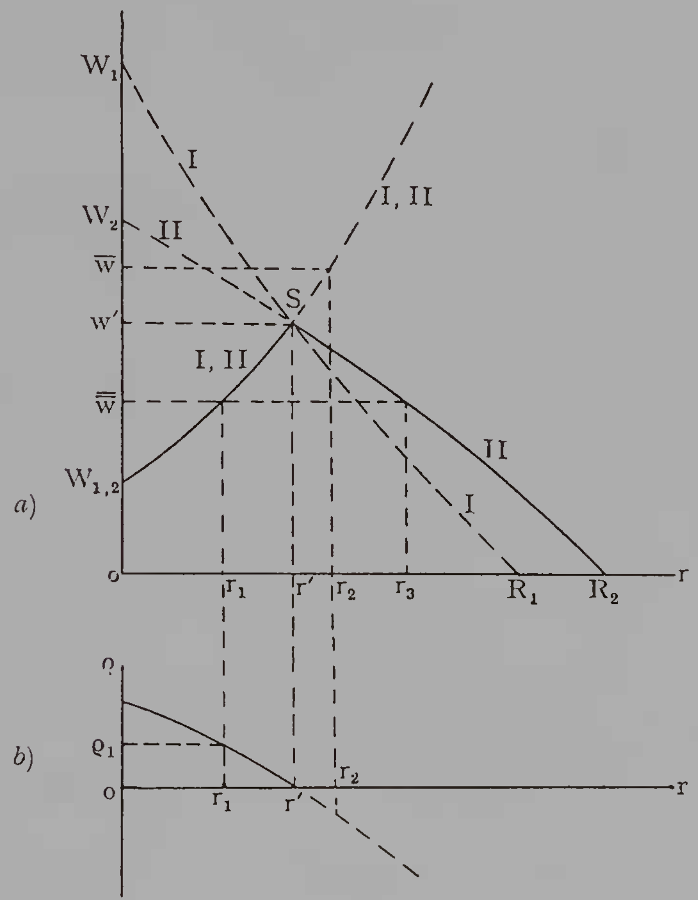


Fig. 9.

([4]) For $r = 0$ equation [25] is:

$$W_2 = \frac{Z^2 (A - A_a) - A_z^2 Z_a}{L_z^2 (A - A_a) + L_a A_z^2}$$

Since the system is supposed to be in a self-replacing state, $W_2 > 0$.

The first derivative of [25] is:

$$\frac{dw}{dr} = \frac{-Z^2 A L_a A_z^2 - A_z^2 Z_a (1+r) \{L_z^2 [2A - A_a (1+r)] + (1+r) L_a A_z^2\}}{y_2'^2}$$

— 92 —

In fig. 9b the following function is represented:

[26] $$\rho(r) = \frac{A(L_z^1 Z^2 - L_z^2 Z^1) + [Z^2(L_a A_z^1 - L_z^1 A_a) - Z^1(L_a A_z^2 - L_z^2 A_a)](1+r) + Z_a(L_z^2 A_z^1 - L_z^1 A_z^2)(1+r)^2}{\Lambda^2 [L_z^1 A + (1+r)(L_a A_z^1 - L_z^1 A_a)] - \Lambda^1 [L_z^2 A + (1+r)(L_a A_z^2 - L_z^2 A_a)]}$$

Function [26] gives either a raising or a falling rate of rent for an increasing rate of profit.

In fig. 9a relations $w_1(r)$, $w_2(r)$ and $w_{1,2}(r)$ assume the same value for the rate of profit r'. For this value of the rate of profit the two methods for corn production are equally convenient and there can be no rent. Indeed, the rate of rent is zero (see fig. 9b) for this value of the rate of profit (5).

The upward-sloping relation between the wage rate and the rate of profit gives rise to an embarassing situation. Let us consider, firstly, the wage rate as given. For a wage rate equal to $\overline{w}$, for instance, it would seem possible at first glance to produce corn with method I and II jointly and to obtain a rate of profit r_2. But at wage rate $\overline{w}$, with that system of production, there is a negative rent rate (see fig. 9b): therefore solution I, II must be discarded. On the other hand, at wage rate $\overline{w}$, method I is more convenient than method II. Therefore agricultural entrepreneurs will try to introduce the first method causing a land scarcity and will enable landlords to ask for rent. If the wage rate does not change, competition among agricultural entrepreneurs will continue until the rate of profit falls to zero and rents to landlords will be very high. At this point however, no one would be a capitalist and production would stop. We must therefore conclude that w' is the maximum wage rate for the economy.

For some wage rate smaller than w', a second curious situation will rise. Take for instance wage rate $\overline{\overline{w}}$. At that wage, two systems of production can be used. The first one is system I, II, which gives the rate of profit r_1 and a positive rate of rent ρ_1. The second system

---

for values of r included between O and $\frac{2A - A_a}{A_a}$ expression { } is positive therefore $\frac{dw}{dr} < 0$ for r included between O and R, since $R < \frac{A - A_a}{A_a}$ $\left(\text{for } r = \frac{A - A_a}{A_a} \text{ equation [25] gives a value of } w = \frac{-Z_a A}{L_a A_z} < 0\right)$.

(5) The sign of the rate of rent may be deduced by a formula (deduced from equation [26] after some manipulation) formally equal to that written in footnote 6 of the previous paragraph.

Note that $w_2(r)$ and $w_1(r)$ have a switching point for $r = \frac{A - A_a}{A_a}$ and that, for this value of the rate of profit, wages are negative, since $R < \frac{A - A_a}{A_a}$ for $w_2(r)$ and $w_1(r)$.

— 93 —

which can be active, at that wage, is II, which gives a rate of profit r_3 and a zero rate of rent. Therefore, the physical surplus (over the wages) can be distributed to profits and rents in the first case and only to profits in the second. Which one of the two solutions is adopted is probably a question to be studied in relation to the real functioning of the economy outside the « natural » (i.e. the solutions of the production equations) values of the variables.

Let us consider now the case in which the rate of profit is given. For a rate of profit included between zero and r' corn will be produced with method I and II jointly. Indeed, let us take the rate of profit r_1: in such a case method I is more convenient than method II. Entrepreneurs employing method I are able either to get extra profits (or to pay extra wages) or to pay a rent higher than the one paid by entrepreneurs employing method II. Therefore, there is a tendency, due to competition among agricultural entrepreneurs, for the rate of rent to rise to the value ρ_1 and for the wage rate to go down to the value $\bar{\bar{w}}$. For these values of the rates of rent and wage, methods I and II can coexist and corn will be produced with the two methods jointly.

For a rate of profit included between r' and R_2 only method II is used and land will be redundant (system I, II must be discarded because it gives a negative rate of rent for these values of the rate of profit).

We can, therefore, conclude that the wage-profit frontier is $W_{1,2}SR_2$.

APPENDIX

In this numerical appendix some exemples will be shown, to illustrate the cases discussed in the text. For each example the physical productivity $\left(\frac{Z^h}{\Lambda^h}\right)$ of the first method for corn production is assumed to be smaller than the productivity of the second method. The total quantity of corn to be produced is that indicated in the relative paragraph of the text, for every example.

Example 1

This example concerns differential rent. The system of production presented here is similar to the one discussed in section I, §. 2: two alternative methods can be adopted for corn production and only one method is known for industrial production. Let us suppose that the first system, when Λ^1 is the marginal land, is:

$$5\, p_a^{(1)}(1+r) + 0.3\, w^{(1)} = 10\, p_a$$
$$1\, p_a^{(1)}(1+r) + 0.04\, w^{(1)} = 2.1$$

From these equations we can deduce relation $w_1(r)$ and, given a certain rate of profit (and therefore $w^{(1)}$ and $p_a^{(1)}$ too) we can find the value of ρ^2 from the third equation, regarding production on land Λ^2, as follows:

$$2\, p_a^{(1)}(1+r) + 0.5\, w^{(1)} + 1\rho^2 = 6.5$$

— 94 —

On the other hand, when Λ^2 is the marginal land, the wage-profit relation $w_2(r)$ is deduced from the two following equations:

$$5\,p_a^{(2)}(1+r) + 0.3\,w^{(2)} = 10\,p_a^{(2)}$$
$$2\,p_a^{(2)}(1+r) + 0.5\,w^{(2)} = 6.5$$

and the value of the rent rate ρ^1 is calculated from the following equation (where $p_a^{(2)}$ and $w^{(2)}$ are known):

$$1\,p_a^{(2)}(1+r) + 0.04\,w^{(2)} + 1\,\rho^1 = 2.1$$

The main values of the two systems are shown in the figure.

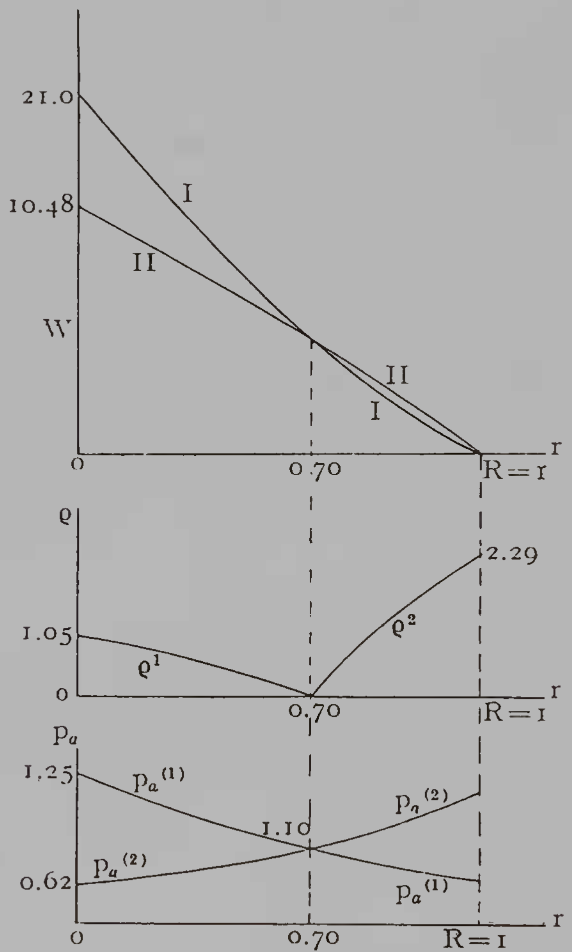


Fig. A.

Example 2

Let us now consider rent on land of a single quality. The example here presented refers to the first case discussed in section II, §. 2. The system for the production of corn with two methods is:

$$5\,p_a(1+r) + 0.3\,w = 10\,p_a$$
$$1\,p_a(1+r) + 0.04\,w + 1\,\rho = 2.1$$
$$2\,p_a(1+r) + 0.5\,w + 1\,\rho = 6.5$$

The more significant values, for this system, are shown in the table below.

— 95 —

r	w	ρ	p_a
0.00	8.46	1.25	0.50
0.10	8.25	1.16	0.55
0.20	7.99	1.06	0.59
0.30	7.70	0.93	0.65
0.40	7.33	0.78	0.73
0.50	6.87	0.58	0.82
0.60	6.28	0.34	0.94
0.70	5.50	0.01	1.09
0.71	5.40	— 0.02	1.11
0.80	4.40	— 0.45	1.31
0.90	2.75	— 1.14	1.64
1.00	0.00	— 2.29	2.19

From the first and the second equation we can deduce relation $w_1(r)$. The main values of this system are:

r	w	p_a
0.00	21.00	1.25
0.50	9.54	1.14
0.70	5.52	1.10
0.71	5.33	1.10
0.90	1.77	1.06
1.00	0.00	1.04

Finally, from the first and tne third equation we can deduce relation $w_2(r)$. Given relation $w_2(r)$ the values of w and p_a are calculated as follows:

r	w	p_a
0.00	10.48	0.62
0.50	7.55	0.90
0.70	5.50	1.10
0.71	5.38	1.11
0.90	2.33	1.40
1.00	0.00	1.62

Therefore, for values of the profit rate falling between 0 and 0.70 corn is raised jointly by the two methods known to us and there will be a positive rate of rent For rates of profit falling between 0.70 and 1 corn is raised only by the second method and there will be no rent.

The behaviour of these functions is shown in fig. 6 in the text.

— 96 —

EXAMPLE 3

Let us now consider a case of rent on land of a single quality, with a profit rate and a rent rate increasing simultaneously.
The system for corn production with the two methods is:

$$5\,p_a\,(1 + r) + 0.3\,w = 10\,p_a$$
$$1\,p_a\,(1 + r) + 0.2\;\;w + 1\,\rho = 6.5$$
$$3\,p_a\,(1 + r) + 0.04\,w + 1\,\rho = 7$$

The pecularity of this system is that it gives positive values of the wage rate (and positive prices too) only for values of the rate of profit equal or grater than 0.15 (approximately). The tabulated values are:

r	w	ρ	p_a
0.00	— 12.49	9.74	— 0.74
0.14	— 537.31	156.69	— 37.48
0.15	212.52	— 53.25	15.00
0.20	25.00	— 0.75	1.87
0.21	21.01	0.36	1.59
0.40	4.16	5.08	0.41
0.60	1.56	5.81	0.26
0.80	0.54	6.09	0.16
1.00	0.00	6.24	0.12

The second system of equations related to the use of the first method for corn production only, gives the following values:

r	w	p_a
0.00	25.00	1.49
0.20	22.41	1.68
0.21	22.26	1.69
0.40	19.11	1.91
0.60	14.77	2.21
0.80	8.78	2.63
1.00	0.00	3.24

— 97 —

The third system of equations, with the second method for corn production, gives:

r	w	p_a
0.00	31.81	1.90
0.20	22.58	1.69
0.21	22.17	1.68
0.40	15.21	1.52
0.60	9.21	1.38
0.80	4.21	1.26
1.00	0.00	1.66

The three systems can be illustrated in the same diagram:

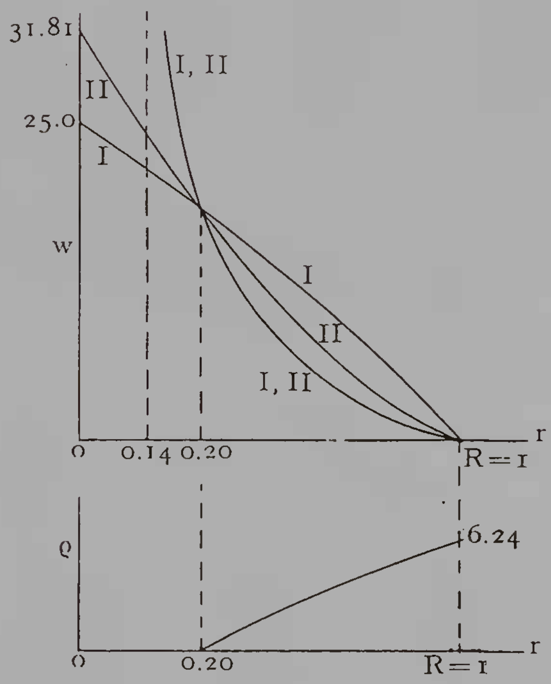


Fig. B.

There is no maximum wage for system I, II. But W'_{II} = 31.81 is the maximum wage rate for the economy, since for rates of profit falling between 0 and 0.20 system II is in use. For rates of profit falling between 0.20 and R = 1 , system I, II is in use and there will be a positive rent rate.

— 98 —

EXAMPLE 4

This example refers to the case discussed in section II, §. 3, in which $Z_a > 0$, but the coefficients of production are such as to give a normal wage-profit relation. The system for joint production is:

$$(5\,p_a + 2)(1 + r) + 0.3\,w = 10\,p_a$$
$$0.1\,p_a(1 + r) + 0.1\,w + 1\,\rho = 3.3$$
$$0.2\,p_a(1 + r) + 0.6\,w + 1\,\rho = 17$$

The values of the variables are:

r	w	ρ	p_a
0.00	26.99	0.39	2.01
0.25	26.69	0.27	2.80
0.52	26.02	0.01	4.51
0.53	25.98	— 0.005	4.61
0.75	24.37	— 0.65	8.64
0.98	5.35	— 8.25	55.66
0.99	— 1.26	— 10.90	72.01

The system concerning corn cultivation with the first method is:

$$(5\,p_a + 2)(1 + r) + 0.3\,w = 10\,p_a$$
$$0.1\,p_a(1 + r) + 0.1\,w = 3.3.$$

and gives the following values

r	w	p_a
0.00	30.75	2.24
0.25	29.24	3.00
0.52	26.11	4.53
0.53	25.94	4.61
0.75	19.78	7.54
0.95	0.77	16.52
0.96	— 1.37	17.53

The system for corn production with the second method only is:

$$(5\,p_a + 2)(1 + r) + 0.3\,w = 10\,p_a$$
$$0.2\,p_a(1 + r) + 0.6\,w = 17$$

— 99 —

and the tabulated values of the variables are:

r	w	p_a
0.00	27.64	2.05
0.25	27.15	2.83
0.52	26.04	4.52
0.53	25.97	4.61
0.75	23.42	8.42
0.98	0.73	41.81
0.99	— 4.91	50.11

Therefore, for rates of profit falling between 0 and 0.52 system I, II is in use and there will be a positive rent. For rates of profit falling between 0.53 and 0.98 only the second method for corn production is adopted and there will be no rent.

The three wage-profit relations, together with the rent-profit relation, behave as follows:

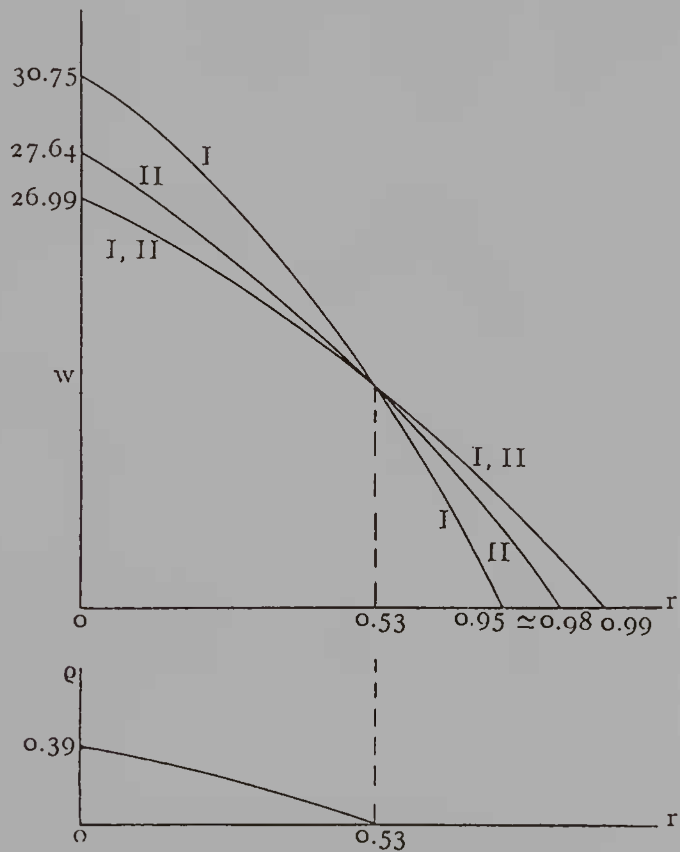


Fig. C.

Example 5

This last example refers to the case in which an anomalous wage-profit relation, as discussed in section II, §. 3, comes out.

The system for joint production is:

$$(5\,p_a + 2)\,(1 + r) + 0.3\,w = 10\,p_a$$
$$3\,p_a\,(1 + r) + 0.1\,w + 1\,\rho = 3$$
$$0.2\,p_a\,(1 + r) + 6\,w + 1\,\rho = 7$$

The tabulated values of the variables are:

r	w	ρ	p_a
0.00	0.89	1.54	0.45
0.10	0.96	1.07	0.55
0.25	1.12	0.04	0.75
0.26	1.14	— 0.03	0.77
0.50	1.67	— 3.47	1.40
0.75	3.75	— 16.80	3.70
0.94	158.61	— 1011.25	171.54

The system of production with the first method only is:

$$(5\,p_a + 2)\,(1 + r) + 0.3\,w = 10\,p_a$$
$$3\,p_a\,(1 + r) + 0.1\,w = 3$$

and gives the following values

r	w	p_a
0.00	6.42	0.785
0.10	4.33	0.777
0.25	1.25	0.766
0.26	1.04	0.765
0.31	0.03	0.762
0.32	— 0.16	0.761

The third system for corn production with the second method only is:

$$(5\,p_a + 2)\,(1 + r) + 0.3\,w = 10\,p_a$$
$$0.2\,p_a\,(1 + r) + 6\,w = 7$$

— 101 —

and gives the following values:

r	w	p_a
0.00	1.15	0.46
0.10	1.14	0.56
0.25	1.135	0.75
0.26	1.134	0.77
0.50	1.10	1.33
0.75	0.98	3.03
0.95	0.14	15.76
0.96	— 0.10	19.44

For the economy represented by these systems there will be a rising rate of wage until a rate of profit equal to 0.26. For higher rates of profit (up to 0.95) method II only is adopted and there will be no rent.

The graphical representation of such a case is similar to the one illustrated in fig. 9 in the text.

[16]

CAPITALISM, SOCIALISM AND STEADY GROWTH [1]

I. Introduction

The purpose of this paper is that of considering the choice of production techniques from the point of view of both the capitalist entrepreneur maximising the present value of his firm's assets at a given interest rate and the socialist planner maximising the consumption per head associated with the maintenance of a given growth rate.

A model of production is set up, in which output is made of a versatile consumption and production good, called *putty*, and of the *machines* which are made of putty and are necessary to assist labour in order to produce putty. It is assumed that technical choice is irreversible, *i.e.*, that putty is moulded and baked into clay machines of given specifications, which cannot be turned back into putty or into machines of different specifications. Also, their use is not affected by technical progress, which improves the design of new machines but not the operation of those already constructed.

This framework, which Phelps first named "putty-clay," [2] has been widely used in recent economic literature.[3] This paper, however, differs from other putty-clay models in that it does *not* contain two customary assumptions, namely that:

(i) the process of transforming this versatile consumption-production good into durable machines is costless , no labour is needed to mould and bake putty into clay, and

(ii) putty is turned into clay-machines instantaneously, so that there are no gestation lags of investment. Both assumptions, as we shall see, reduce significantly the scope of technical choice.

The first assumption, that the transformation of putty into clay is costless, is necessary to keep a putty-clay model in the realm of a one-commodity world. Only in this case can gross investment be measured simply by the

[1] Acknowledgments are due to Maurice Dobb, Piero Garegnani, Richard Goodwin, Malcolm MacCallum, Joan Robinson, Luigi Spaventa and Piero Sraffa for helpful comments and criticisms on an earlier draft of this paper. Responsibility for any error, needless to say, rests solely with the writer.

[2] E. S. Phelps, "Substitution, Fixed Proportions, Growth and Distribution," *International Economic Review*, September 1963.

[3] L. Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: a Synthesis," *Econometrica*, April 1959; W. E. G. Salter, *Productivity and Technical Change* (Cambridge, 1960); R. M. Solow, "Substitution and Fixed Proportions in the Theory of Capital," *Review of Economic Studies*, April 1966; M. C. Kemp and P. C. Thanh, "On a Class of Growth Models," *Econometrica*, April 1966; R. M. Solow, J. Tobin, C. C. von Weizsäcker and M. Yaari, "Neoclassical Growth with Fixed Factor Proportions," *Review of Economic Studies*, April 1966; C. J. Bliss, "On Putty-clay," *Review of Economic Studies*, April 1968.

amount of putty which is turned into clay in each period. If moulding and baking putty into clay requires labour the value of a new machine expressed in terms of putty depends on the interest rate (or the wage-rate). *Gross* output will be made up of that part of putty which is actually devoted to consumption *plus* the output of machines; in addition to the sector producing putty, one needs as many other sectors as there are units of time—in the course of the gestation period of machines—during which labour is needed to process putty into machines. To measure *net* output a number of other sectors are needed, in addition to the putty-producing sector, equal to the number of time units into which the lifetime of a machine can be broken, from the beginning of its construction to the end of its lifetime, because each machine at each different stage of its construction or its operation is a different commodity. We can look at the production process either as joint production of putty and machines or as joint production of dated putty. In this system, as Professor Kaldor once put it, " the inputs of different dates jointly produce the outputs of different dates; and it is impossible to separate out the contribution to the output of different dates of the input of a single date." [1] Output per head—whether gross or net—associated with a given technique would then depend both on the rate of interest—determining the price of each machine in terms of putty—and the growth rate, determining the relative proportion of putty and machines of all kind in total output. The assumption of the costless transformation of putty into clay and the use of *gross* measures evade this fundamental issue of capital theory.

The second assumption, of no gestation period of investment, which is also customary in putty-clay models, eliminates one of the possible dimensions of technical choice, namely, the possibility of a trade-off between the length of the gestation period and the durability of fixed equipment.[2] Both assumptions, as we shall see, are relevant to the problem of " reswitching " of techniques, *i.e.*, the eligibility of the same technique at more than one level or range of the interest rate, with other techniques being eligible at intermediate levels.[3]

[1] N. Kaldor, " The Controversy on the Theory of Capital," *Econometrica*, July 1937, reprinted in *Essays on Value and Distribution* (1960), p. 159.

[2] A. Bhaduri has investigated this aspect of technical choice in a simple case, in : " An Aspect of Project-selection: Durability *vs.* Construction-period," ECONOMIC JOURNAL, June 1968. He finds that " on economic grounds (other things being equal) one may expect a combination of shorter durability and shorter construction period to be more advantageous in a fast growing economy " (p. 346). Here we shall treat gestation and durability more generally, as being only a partial aspect of technical choice—and not necessarily directly related—without the " other things being equal " assumption.

[3] This phenomenon was first noticed in the modern literature by Joan Robinson, Champernowne and Sraffa (J. Robinson, " The Production Function and the Theory of Capital," *Review of Economic Studies*, 1953; *The Accumulation of Capital* (London: Macmillan, 1956); D. G. Champernowne, " The Production Function and the Theory of Capital: a Comment," *Review of Economic Studies*, 1953; P. Sraffa, *Production of Commodities by Means of Commodities, Prelude to a Critique of Economic*

Neither assumption is made in this paper. A more flexible model will be used instead, which takes into account the labour cost of investment, and the gestation and durability of investment, and is designed to handle production techniques characterised by any possible time profile of output and inputs.

Within this framework conditions for reswitching of techniques are stated, and the problem is shown to be relevant both to the capitalist firm and the socialist planner. A version of the golden rule of accumulation is stated, with a second-best proposition. It is shown that the relevance of the reswitching phenomenon is not affected by technical progress. Relative prices of machines and consumption goods are introduced, and the conditions for macroeconomic equilibrium are examined under both capitalism and socialism. In the context of the model the concept of capital is shown to be dispensable under socialism.

II. Assumptions

There is a versatile commodity, putty, which can be either consumed directly or turned into machines by an irreversible process requiring labour. Time is divided into periods of equal length. Putty is perishable and lasts for one period only, unless it is turned into clay. Clay-machines last for more than one period; their durability depends on their shape, the amount and the time pattern of labour and putty which has gone into their making.

Putty is produced by labour and machines. Labour is homogeneous. The technical specifications of machines, *i.e.*, the pattern of the time flow of inputs and outputs associated with them, differ and cannot be altered after their construction. A " technique of production " is represented by a time-flow of putty-output, in which the putty to be moulded and baked into durable machines appears with the negative sign, and a time-flow of labour inputs. The sequence of the time pattern of putty-output is given by $\{a_i\}$, where $a_i \leqslant 0$ for $i = 0, 1, \ldots k-1$ is the amount of putty which is needed initially to be handed over to the workers making machines during period i (the making of a machine can take more than one period; if one single period is needed, $k = 1$; if putty is being produced by labour only, then $k = 0$); $a_k > 0$, $a_i \geqslant 0$ for $i = k+1, \ldots n$ is the putty which is produced thereafter, during each of the subsequent $n-k+1$ periods. We assume that $\sum_{i=0}^{n} a_i > 0$, *i.e.*, total net putty output over the time of operation is strictly positive. The sequence of the time pattern of labour inputs required first to make machines, then to operate them to produce the flow

Theory (Cambridge University Press, 1960), and has been widely debated in a series of papers in the *Quarterly Journal of Economics*, October 1966. See also G. C. Harcourt, " Some Cambridge Controversies in the Theory of Capital," *Journal of Economic Literature*, June 1969.

of putty output, is given by $\{l_i\}$, where $l_0 > 0$ (because labour is always required to start the process), $l_i \geqslant 0$ for $i = 1; 2, \ldots n$. We also assume that l_n and a_n are both positive. There are constant returns to scale. The scale of a technique of production is taken so that $l_0 = 1$. Any convex combination of two techniques is also a technique, but the number of techniques which cannot be expressed as a convex combination of other techniques is finite. The length of k and n is not necessarily the same for all techniques. If a process does not have to be operated to the nth period, but can be stopped after a number of periods $m < n$, each length of operation of the same process is regarded as a separate process. We neglect "inferior" techniques, *i.e.*, such that they give an amount of output at some period lower than another technique, without having a higher output at some other period, and/or a lower labour input at the same or some other period.

We shall consider the full-employment growth of economies with access to this kind of technology, under institutional conditions corresponding to textbook capitalism, centralised and decentralised socialism. In all systems production is organised in productive units called firms, by managers who are all equally efficient. In each period total labour supply is given, and growing at a steady rate λ, $\lambda > -1$. Labour is hired by firms at a real wage w per man per period, paid at the end of the period. Managers are homogenous with the rest of the working force, and the input of their labour is included in the labour coefficients l_i. Economic systems differ in three respects: property relations, market conditions and criteria for technical choice.

Under centralised socialism physical productive assets belong to the State, which appropriates whatever is produced in excess of the payment of wages. It is a monopsonist in the labour market, and fixes the wage-rate w, to which labour supply is inelastic. Firms are simply administrative units, managers are state officers who are ordered to use the technique chosen by the central planner, and receive the necessary material inputs and wage fund (in excess of their current production of putty) free of charge as grants from the State.[1] Among the production techniques available, the central planner selects the technique maximising the rate of consumption per head associated with the maintenance of full-employment steady growth.

Under decentralised socialism physical productive assets belong to state firms. Firms have access to a perfectly competitive labour market, and have infinite power of borrowing and lending putty from and to the State, at a rate of interest r fixed by the State. They have built their assets by borrowing from the State in the past, they appropriate current output and pay wages and interest out of it. Among the production techniques available,

[1] Central fixing of the wage-rate, free investment funds granted from the state budget, central choice of production techniques, administrative orders to the managers of state firms: these are aspects typical of the pre-war Soviet planning system.

they select the technique maximising the present value of their assets at the ruling interest rate.[1] The socialist planner will still wish the technique maximising consumption per head to be chosen, but the only way he can affect technical choice is by choosing the interest rate r, which is the basis of the decisions of state managers.

Under capitalism, physical productive assets belong to individual capitalists, either directly or through shareholding. Firms have access to a perfectly competitive labour market, and have infinite power of borrowing and lending putty at a rate of interest r. Capitalists appropriate the excess of output over what is needed to pay managers and workers the competitive wage, consume part of it and accumulate the rest. Among the production techniques available, the one which maximises the present value of the assets of capitalists at the ruling interest rate is chosen.

Both under capitalism and decentralised socialism macroeconomic equilibrium requires that the production of putty in excess of current consumption requirements should be equal to the material input requirements in the construction of machines. The conditions for equilibrium will be examined in the next sections; we can imagine, provisionally, that the economy in question is connected with a perfect international capital market.

II. The "Wage-interest" Frontier

We shall first consider the implications of the present-value maximisation criterion for technical choice.

Suppose there is one technique only, and no technical progress. The present value v of starting a unit scale process, $\{a_i\}$, $\{l_i\}$, is given by

$$(1) \qquad v = \sum_{i=0}^{n} (a_i - wl_i)(1 + r)^{-i}.$$

Since the labour market is competitive, as long as v is positive workers will be successful in demanding higher wages, from firms competing with each other trying to get hold of labour. Equilibrium in the labour market requires that

$$(2) \qquad v = 0.$$

[1] These characteristics can be found, for instance, in the Czechoslovak economy in 1967. According to the documents of the 1967 economic reforms, wage guidelines were fixed centrally, but managers could pay additional bonuses to workers, out of an enterprise fund made of retained profits, subject to the payment of a tax on the wage fund, called "stabilisation" tax. See "General Guidelines for Enterprise Operation, Valid from January 1, 1967," in *New Trends in the Czechoslovak Economy*, Booklet No. 6, September 1966. The present value criterion for investment choice was introduced in April 1967 by the State Commission for Technology, *Zásady hodnocení ekonomické efektivnosti investic* (Criteria for the assessment of economic effectiveness of investment), Č.j. 16.653/42/67. See D. M. Nuti, "Investment Reforms in Czechoslovakia," *Soviet Studies*, January 1970.

At each level of the interest rate there is, for a given technique, a maximum wage-rate which firms, performing lending and borrowing operations, can afford to pay to workers and make no loss. This is given by the following equation, obtained from (1) and (2):

$$w = \frac{\sum_{i=0}^{n} a_i(1+r)^{-i}}{\sum_{i=0}^{n} l_i(1+r)^{-i}} \tag{3}$$

This we call the " wage–interest frontier." (The general form of this function, $w = w(r)$, is discussed in the mathematical appendix.) The function has the following properties:

(i) for $r = 0$, $w = \sum_{i=0}^{n} a_i / \sum_{i=0}^{n} l_i > 0$

(ii) there is only one value of r, r^*, for which $w(r) = 0$ because $\sum_{i=0}^{n} l_i(1+r)^{-i}$ is always positive, and because there is only one inversion of sign in the coefficients of the polynomial at the numerator.[1]

From (i) and (ii) it follows that $w(r) > 0$ for $0 \leqslant r < r^*$. (iii) the sign of the first derivative of $w(r)$ is negative for $r = r^*$, but for $0 < r < r^*$ does not have to be negative throughout, and the graph of $w(r)$ may present " bumps." The maximum number of bumps is shown in the appendix to be given by the number of alternations of sign of $\left(\frac{a_{i+1}}{l_{i+1}} - \frac{a_i}{l_i}\right)$, for $i = k, \ldots n$. Bumps therefore might occur if output per man fluctuates from the kth period onwards, for instance, if machines require periodical repairs and spare parts are made out of current output (a_i could even become negative for some $i > k$ if repairs requirements exceed current output, but we have assumed that this is never the case). The economic meaning of the bump is that, over some range of the rate of interest, a firm is a borrower in some periods and a lender in some other periods, and it gains from an increase of the interest rate as a lender more than it loses as a borrower, so that it is able to pay a higher wage-rate if it can perform lending–borrowing operations at a higher interest rate. The presence of bumps, however, is not essential to the following argument.

(iv) The only cases in which the $w(r)$ function is a straight line are ones in which $l_0 = 0$. This will never be the case under our assumptions, because we always have $l_0 > 0$.

Possible graphs of equation (3) are given in Fig. 1.

[1] The number of positive real roots of a real polynomial is equal to the number q of its variations of sign—after having suppressed all terms having zeros as coefficients—or is less than q by a positive even integer.

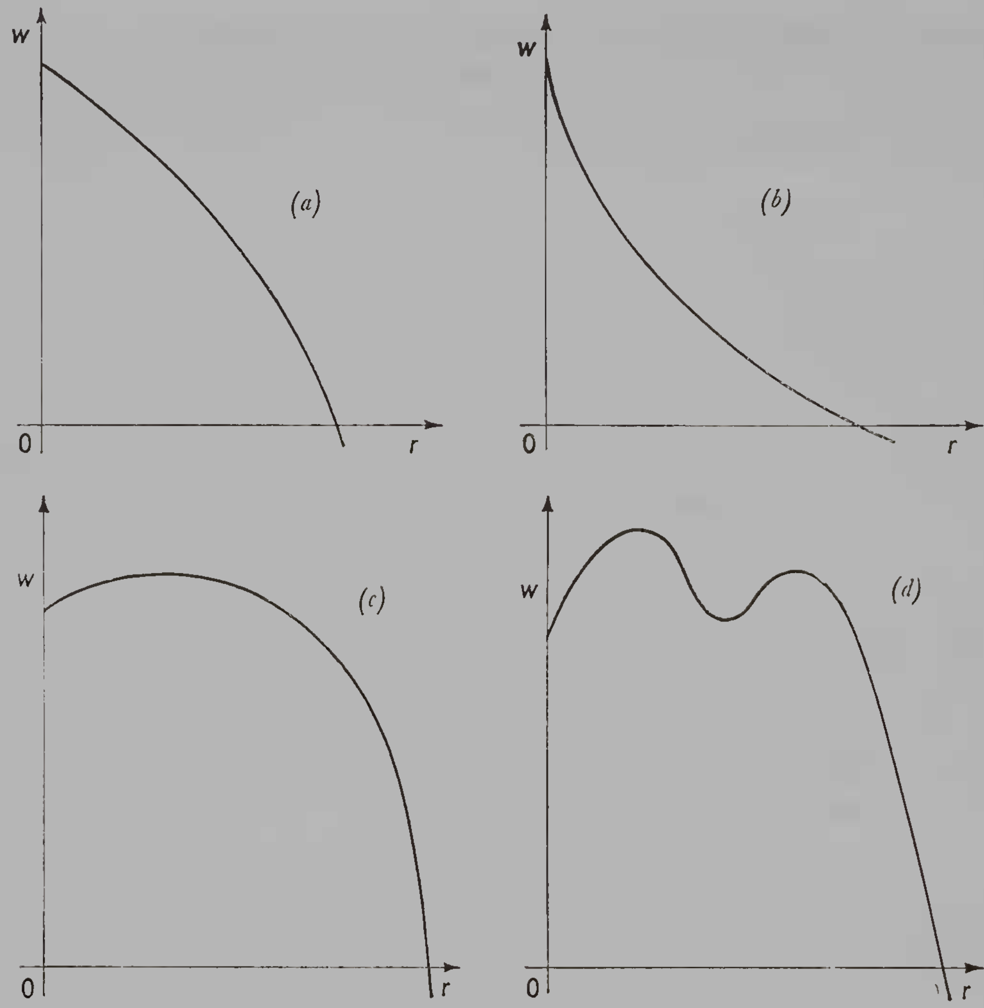


FIG. 1

If a given process does not have to be operated to the nth period but can be stopped before at no additional expense we draw the wage–interest frontier for each length of operation T such that $k \leqslant T \leqslant n$, and superimpose them on the same diagram.[1] Some of them might be inferior. For instance, if output per man is constant after the machine is built, *i.e.*, $\frac{a_i}{l_i} = \bar{a}$ for $i \geqslant k$, any length of operation $T < n$ will give a lower wage-rate than $T = n$ at all values of the rate of interest. If, however, output per man varies over the operation of a machine it might happen that different lengths of operation will be best over different ranges of the interest rate. If the wage frontier has bumps this procedure will smooth the bumps out of the external boundary of the frontiers.[2] If different lengths of operation of

[1] Of course there is no point is considering $T < k$, because $\sum_{i=0}^{T} a_i \leqslant 0$ for $T < k$, and at non-negative interest rates the wage would be negative.

[2] Choosing the length of operation T might not always be possible, for instance, if putty is mined in open-cast mines requiring the replacement of topsoil with relatively large labour expenses towards the end of the operation of the process.

a technique appear in the outer boundary of its wage frontiers the optimum economic lifetime of plant is shown to depend on the interest rate.

If we perform the same operation for all techniques of production available, and superimpose all the $w = w(r)$ functions in the same diagram, we obtain a picture whose outer boundary gives the maximum wage-rate which firms confronted with a given range of techniques can afford to pay, given the rate of interest at which they can undertake lending and borrowing operations. Throughout this paper by $w(r)$ we shall always indicate this outer boundary, which is illustrated in Fig. 2.

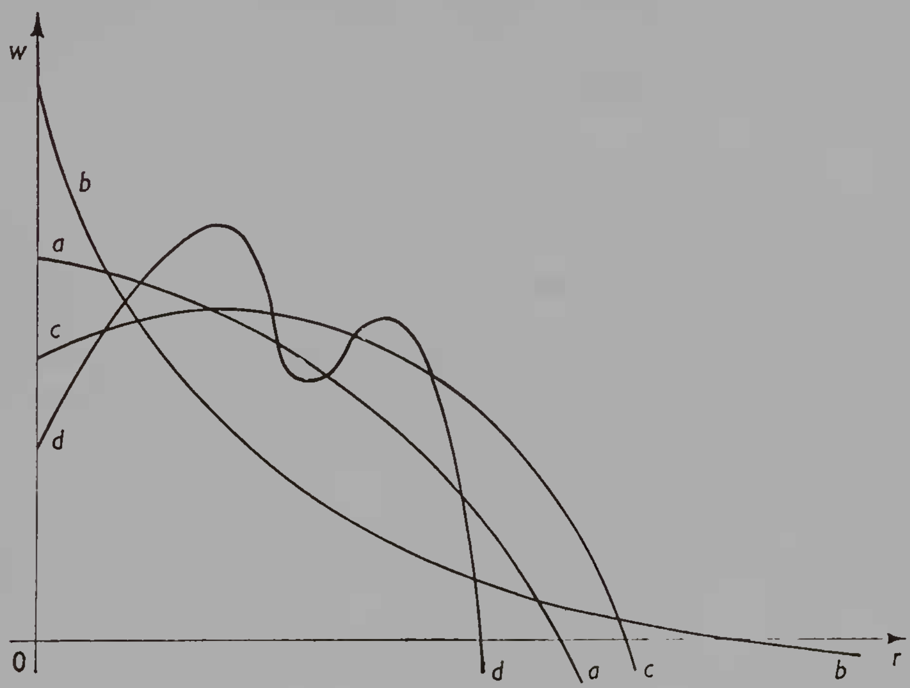


FIG. 2

It might be impossible to rank techniques of production so that each technique is associated with a single value or range of values of the interest rate. Reswitching of techniques might be observed in economies with access to the same technology and different values of the interest rate: the same technique might be in use at two different values of r, with another technique used at intermediate values of r. If there are two techniques, A and B, reswitching means that A affords the same wage-rate as B at more than one level of the interest rate. Suppose technique A is given by (a_{Ai}, l_{Ai}), where

$$a_{Ai} \leqslant 0 \text{ for } i = 0,1, \ldots k_A - 1$$
$$a_{Ai} \geqslant 0 \text{ for } i = k_A, \ldots n_A$$
$$l_{Ai} \geqslant 0 \text{ for } i = 0, \ldots n_A$$

and technique B is given by (a_{Bi}, l_{Bi}), and $k_A \gtreqless k_B$, $n_A \gtreqless n_B$. Reswitching will occur if the equation

$$(4) \qquad \frac{\sum_{i=0}^{n_A} a_{Ai}(1+r)^{-i}}{\sum_{i=0}^{n_A} l_{Ai}(1+r)^{-i}} - \frac{\sum_{i=0}^{n_B} a_{Bi}(1+r)^{-i}}{\sum_{i=0}^{n_B} l_{Bi}(1+r)^{-i}} = 0$$

has more than one positive root. This condition can be rewritten as

$$(5) \quad \sum_{i=0}^{n_B} l_{Bi}(1+r)^{-i} \sum_{i=0}^{n_A} a_{Ai}(1+r)^{-i} - \sum_{i=0}^{n_A} l_{Ai}(1+r)^{-i} \sum_{i=0}^{n_B} a_{Bi}(1+r)^{-i} = 0$$

having more than one positive root. There is no reason whatsoever to assume that this is not the case on grounds of realism. Suppose that the two techniques are such that $n_A = n_B$ and $l_{Ai} = l_{Bi}$ for all $i = 0, 1, \ldots n$. The condition for reswitching becomes

$$(6) \qquad \sum_{i=0}^{n} (a_{Ai} - a_{Bi})(1+r)^{-i} = 0$$

having more than one positive root. The necessary (but not sufficient) condition for this being the case is that the sign of $(a_{Ai} - a_{Bi})$ should alternate more than once: there is nothing extravagant in assuming that output (investment counting as negative output) with one technique is higher in two periods and lower in an intermediate period, with respect to another technique, as in Fig. 3 below.

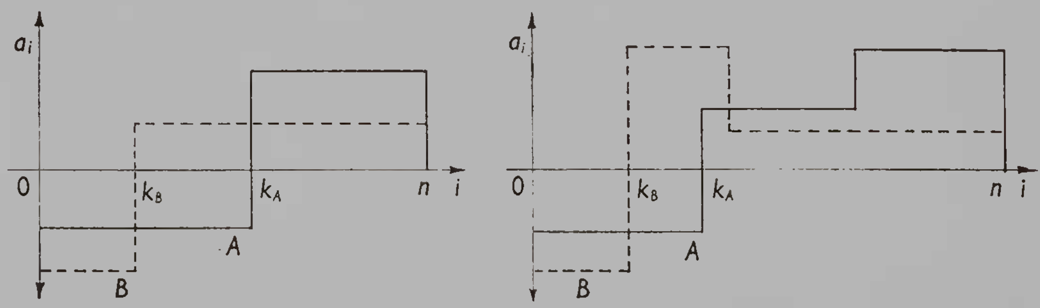


Fig. 3

The actual number of roots (and therefore of switching points) can be found by using Sturm's theorem.[1] When reswitching occurs, the available

[1] Let $f(x)$ be a polynomial with real coefficients such that $f(x) = 0$ has no multiple roots. Construct the identities

$$c_0 f = q_1 f' - f_2, \; c_1 f' = q_2 f_2 - f_3, \; c_2 f_2 = q_3 f_3 - f_4,$$
$$\cdots\cdots\cdots\cdots$$
$$c_{k-2} f_{k-2} = q_{k-1} f_{k-1} - f_k,$$

where q_r/c_{r-1} is the quotient of the division f_{r-1}/f_r; f_k is a constant $\neq 0$, and each f_r is of degree one less than its predecessor. Let a and b be real numbers neither of which is a root of $f(x) = 0$, while $a < b$. Then the number of real roots between a and b of $f(x) = 0$ is the excess of the number of variations of sign in the chain

$$f(x), f'(x), f_2(x), \ldots, f_{k-1}(x), f_k$$

for $x = a$ over the number of their variations of sign for $x = b$. Terms which vanish are to be discarded before counting the variations of sign.

blueprints cannot be so ordered in a book that at a higher interest rate a higher numbered page contains the " best " technique, unless the same blueprint can be inserted more than once. It should be noted that the actual number of reswitching points between the wage frontiers of two techniques is totally uninteresting: in a sense, we can say that the greater the number of reswitching points, the closer the two techniques can be considered to be, and therefore the less important the fact of reswitching. A better measure, however loose, of the importance of reswitching can be given by the maximum difference between the wage-rates afforded by the two techniques at the same rate of interest, because this is a measure of the maximum inefficiency which can result from a wrong choice of techniques (or otherwise some other statistics of the distribution of such differences, taken with the positive sign: $|w_A(r) - w_B(r)|$).

IV. The " Consumption-growth " Frontier

We shall now look at what determines, under the technical conditions already described, the level of consumption per head at different alternative steady growth rates, and its relation with the wage-frontier.

Suppose there is only one technique available, the number of projects (of unit scale) started in each period has been increasing at a constant rate g per period in the last n periods, and the amount of labour currently employed on projects just started is L_t. The number of projects started this period therefore is given by $L_t/l_0 = L_t$; the number of projects started during the period $t-1$ is equal to $L_t(1 + g)^{-1}$, and in general the number of projects started at time $t-i$ is equal to $L_t(1 + g)^{-i}$. A project started at time $t-i$ will require l_i units of labour and will be associated with a_i units of output (or $-a_i$ units of investment, if $i < k$). Current employment on projects started at time $t - i$, L_{t-i}, is therefore given by equation (7):

$$L_{t-i} = L_t(1 + g)^{-i}l_i, \quad i = 0,1, \ldots n \tag{7}$$

From this we can now determine total employment, N; total gross putty output, X; total putty needed as a material to make machines, J; and consumption, C. They are given by the following equations:

$$N_t = \sum_{i=0}^{n} l_i(1 + g)^{-i} . L_t \tag{8}$$

$$X_t = \sum_{i=k}^{n} a_i(1 + g)^{-i} . L_t \tag{9}$$

$$J_t = -\sum_{i=0}^{k-1} a_i(1 + g)^{-i} . L_t \tag{10}$$

$$C_t = X_t - J_t = \sum_{i=0}^{n} a_i(1 + g)^{-i} . L_t \tag{11}$$

From equations (8), (9) and (11) we can express gross putty output per head, $x = X/N$, and consumption per head $c = C/N$ as a function of the growth rate of investment:

$$x = \frac{\sum_{i=k}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}} \tag{12}$$

$$c = \frac{\sum_{i=0}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}} \tag{13}$$

Consumption and gross output of putty per head appear therefore to depend solely on the steady growth rate of investment, which will be also the growth rate of the whole economy (as long as investment has been growing at that rate for the last n periods). At full employment (and without technical progress as we have assumed so far) the rate of growth in investment g will have to be equal to the rate of growth of employment λ. Equation (13), expressing consumption per head c as a function of the growth rate g of investment, $c = c(g)$ is exactly identical to equation (3), the wage–interest frontier, with g instead of r and c instead of w. All we have said in relation to equation (3) applies also to equation (13), which we shall call the "consumption-growth" frontier, because each of its points indicates the maximum consumption per head corresponding to a given steady growth rate, and vice-versa, the growth rate (or rates, if there are "bumps") achievable with a given level of consumption per head. This relation holds both in a socialist planned and in a capitalist economy, growing at a steady growth rate. If there is more than one technique, however, only under centrally planned socialism will the technical choice be determined with reference to the consumption per head maintainable at a given growth rate, whereas under capitalism and decentralised socialism maximisation of present value, as we shall see, might lead to the choice of a different technique.

If we draw the graph of equation (13) for all techniques of production available, the outer boundary will give the maximum level of consumption per head which is consistent with each growth rate. The picture is represented in Fig. 4, which looks exactly like Fig. 2, so that we can measure w, c on the vertical axes and r, g on the horizontal axes. We can now draw the functions also for $g < 0$ and for $c(g) < 0$: negative growth rates—unlike negative interest rates—are economically quite plausible, and the properties of a steadily declining economy can be explored. Negative consumable output per head at some growth rate indicates how much steady external aid per head is needed, on top of subsistence real consumption per head, to maintain that growth rate.[1] However, in order to draw conclusions out of this framework, we need to know not only the outer boundary of the frontiers but also the whole network of frontiers and their interweaving. Under capitalism or decentralised socialism, where technical choice is based on the maximisation of present-value criterion, consumable putty-output per head c will be a function both of the interest rate, which determines the technique chosen, and of the rate of growth of investment. Let us call $a_{i,r}$ and $l_{i,r}$ the technical coefficients of the technique selected at an interest rate r. The function expressing consumable output per head as a function of the growth rate and the interest rate, $c = c(r, g)$ can be written as

$$c = \frac{\sum_{i=0}^{n} a_{i,r}(1+g)^{-i}}{\sum_{i=0}^{n} l_{i,r}(1+g)^{-i}} \tag{13'}$$

If the rate of interest differs from the growth rate, in such conditions consumption per head is not necessarily located on the outer boundary of the frontiers. We can now state the following propositions:

(i) All we have said about reswitching of techniques at alternative interest rates applies here to the reswitching of techniques at alternative steady state growth rates. (Hence, the same relation holds between T and g for each technique, as it holds for T and r.) If growth has been efficiently planned by socialist planners, one might find the same, consumption-maximising technique in two economies where investment grows at a different rate, and another technique in a third economy where investment grows at an intermediate rate.[2]

(ii) If the criterion for technical choice is present-value maximisation at a given interest rate, in a competitive labour market, we can state the following version of the " golden rule ": [3] " For a given growth rate of

[1] The maximum number of bumps in the function $c = c(g)$ for $c < 0$ is given by the number of alternations of sign of

$$\left(\frac{a_{i+1}}{l_{i+1}} - \frac{a_i}{l_i}\right) \text{ for } i < k.$$

[2] If the consumption-growth frontier is increasing over a particular range of the growth rate the corresponding growth rates are in a sense inefficient, in that higher growth rates could have been attained, raising consumption per head instead of reducing it. The " bump " in the frontier did not matter for the firm, which had to take the interest rate as given, but matters for the planner to the extent to which he can control the rate of growth of labour supply through immigration and population policy.

[3] This is the mirror image of von Neumann's statement about the conditions to obtain the maximum growth rate corresponding to a given level of consumption per head, in: " A Model of General Equilibrium," *Review of Economic Studies*, 1945. Several versions of this rule have appeared since: see F. H. Hahn and R. C. O. Matthews, " The Theory of Economic Growth: a Survey," ECONOMIC JOURNAL, December 1964. In the context of planned socialist growth the same rule is also stated by M. H. Dobb in *Welfare Economics and the Economics of Socialism* (Cambridge, 1969), Ch. 8.

investment, a sufficient condition for consumption per head to be the highest consistent with such growth rate is that the rate of interest should be equal to the rate of growth of investment. If the number of techniques available is infinite, and there is no reswitching, and the switching points are dense, this is also a necessary condition." From Fig. 4 we can see that for any given value of g, say $\bar{g}$: (*a*) If $r = \bar{g}$, the technique (or techniques if there is a switch point at $\bar{g}$) chosen is that yielding the maximum consumption per head attainable at that growth rate. (*b*) Let us call the switching values of the rates of growth and interest a, b, e and f; if the consumption-maximising technique switches at $g = b < \bar{g}$ and at $g = e > \bar{g}$, then as

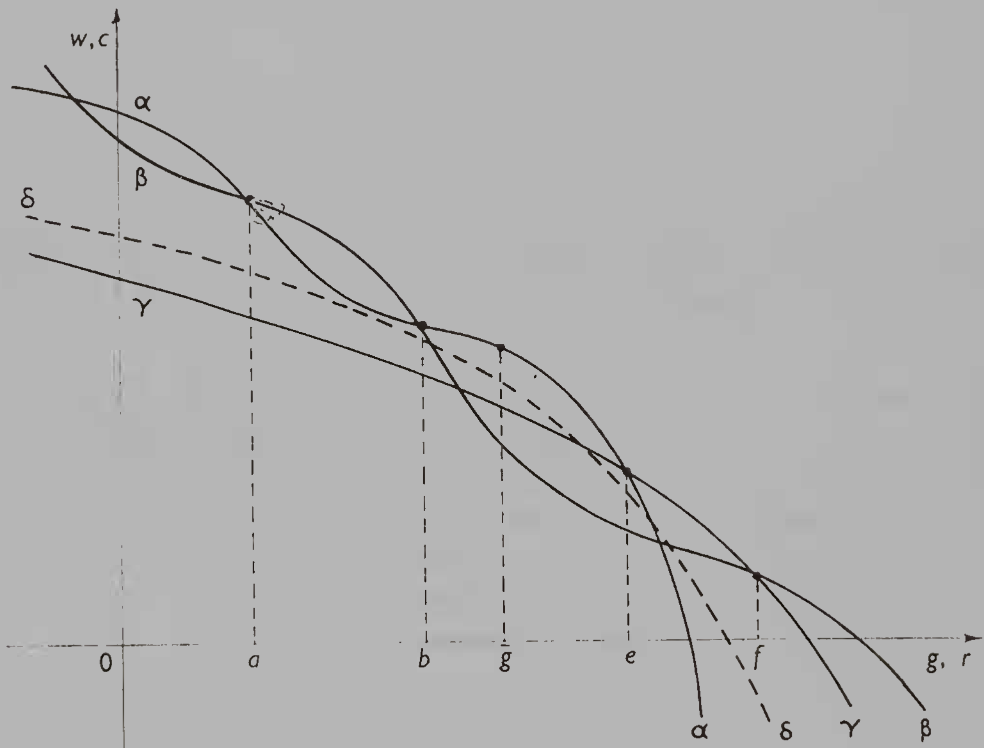


FIG. 4

long as $b < r < e$ the present-value-maximising technique and the consumption-maximising technique will be the same (at $r = b$ or $r = e$ present value could be maximised by linear combinations of two techniques, but this would not necessarily maximise consumable output per head). (*c*) If there is reswitching the technique which maximises consumable output per head at a rate of growth $\bar{g}$ might maximise present value also over some other range of r. In Fig. 4, for instance, the technique maximising consumable output at $\bar{g}$ is also chosen for $0 < r < a$ as well as $b < r < e$. This means that if $a < r < b$ firms can be induced to choose the consumption-maximising technique either by increasing the interest rate, bringing it closer to $\bar{g}$, or by reducing it further and bringing it closer to zero. The difference between g and r, in other words, cannot be taken as a measure of inefficiency. (*d*) Suboptimality can take not only the form of the wrong plant but also of the wrong length of operation of the " right " plant.

(iii) We can also state the following " second-best " proposition (whether or not reswitching occurs). If $r \neq \bar{g}$, consumption per head might be higher for values of r farther away from $\bar{g}$ than for values closer to $\bar{g}$, and if for some reason the ranges of r over which the (consumption-maximising) technique is chosen are unattainable, there will be a range of values of r over which a " second-best " technique will be chosen, yielding the second highest consumption per head at a rate of growth $\bar{g}$ *among the techniques forming the frontier*. In Fig. 4 this is technique γ, which would be chosen over the range $c < r < f$. It appears, however, that, at the rate of growth $\bar{g}$, γ is inferior to a technique δ which does not appear anywhere along the frontier, and will never be chosen at any value of the interest rate. A typical case would be that of the steadily declining economy, where, if the rate of interest is not allowed to be negative, the consumption-maximising technique will never be chosen by firms (unless that technique is also the best at positive growth and interest rates). If wages and prices, however, are expressed in money terms and are expected to change in time at a steady percentage rate p the parameter relevant to technical choice would not be r, but $\left(\frac{1+r}{1+p} - 1\right)$. Even if there are constraints on the values of r, this " deflated " interest rate can be made equal to g, provided expectations can be generated of a steady percentage rate of price increase p such that

$$p = \frac{r - g}{1 + g} \tag{14}$$

The rule for obtaining optimal technical choice in conditions of steady state growth would now become $r = p + (1 + p)g$.

V. Technical Progress

Suppose technical progress takes place in time, exogenous, disembodied and neutral, in that it decreases labour inputs at all stages for all techniques by the same proportion $d < 1$. If the real wage increases at the rate $h = \frac{d}{1 - d}$ the relative profitability of different processes is not altered, and the golden rule remains the same as before. If technical progress is neutral but, as we have have assumed in this model, is " embodied " in machines, which permamently have the input and output characteristics of the time of their construction, and real wages increase at the rate $h = \frac{d}{1 - d}$ while labour inputs steadily decrease from one blue-print book to another at the rate d, the present value of starting a unit scale project at time t, v_t, is given by equation (1′ ″):

$$(1' '') \qquad v_t = \sum_{i=0}^{n} [a_i - w_t(1+h)^i l_i] \,.\, (1+r)^{-i}$$

and the maximum real wage-rate w_t which can be afforded at time t, on the understanding that it must increase at the rate h, is given by putting $v_t = 0$:

$$(3'' ') \qquad w_t = \frac{\sum_{i=0}^{n} a_i(1+r)^{-i}}{\sum_{i=0}^{n} l_i(1+r)^{-i}(1+h)^i}$$

For any state of knowledge at any given time t the real wage-rate will be lower, with respect to the situation without technical progress, if wages are expected to increase for all workers at the rate $h = \frac{d}{1-d}$ as labour inputs are reduced by technical progress at the rate d on machines whose construction is currently beginning. The graph of equation (3″ ′) is similar to that of equation (3), but the ranking of techniques and the number and position of switchpoints will differ at different values of h.

If a given technique does not have to be used to the end of its physical life, occurring in period n, but can be stopped earlier at no extra cost, we can again superimpose in the same diagram the wage–interest frontiers corresponding to different lengths of operation T of that technique, $k \leqslant T \leqslant n$. With wages rising at a rate h, the optimum economic lifetime might differ from n, even if $a_i/l_i = \bar{a}$ for $i \geqslant k$. Its actual length will depend on the interest rate. The same relation holding between T and r will hold also between T and g: given the technical coefficients and their rate of change in time, the best length of operation of a given technique from the point of view of maximisation of consumption per head will depend on the growth rate.

Given two techniques A and B, as described in Section III, the conditions for reswitching between them, which in the absence of technical progress was given by equation (5) having more than one positive root, becomes now that equation

$$(5') \qquad \sum_{i=0}^{nB} l_{Bi}(1+r)^{-i}(1+h)^i \sum_{i=0}^{nA} a_{Ai}(1+r)^{-i} - \sum_{i=0}^{nA} l_{Ai}(1+r)^{-i}(1+h)^i \sum_{i=0}^{nB} a_{Bi}(1+r)^{-i} = 0$$

should have more than one positive root. Again, there is no reason whatsoever to assume that this is not the case on grounds of realism. Suppose that the two techniques are such that $n_A = n_B$ and $l_{Ai} = l_{Bi}$ for all $i = 0,1, \ldots n$. The condition for reswitching is still expressed by

$$(6) \qquad \sum_{i=0}^{n} (a_{Ai} - a_{Bi})(1+r)^{-i} = 0$$

having more than one positive root. This is exactly as in the case *without* technical progress: at each value of r such that $0 \leqslant r < r^*$ the real wage-rate, corresponding to a given technique if technical progress is expected to take place, will be lower, of course, than if technical progress were not expected, but r^* for each technique, and the switching values of r between techniques, will be the same. Suppose now that the two techniques A and B are such that $n_A = n_B$ and $a_{Ai} = a_{Bi}$ for $i = 0,1, \ldots n$, but differ for more than one labour coefficient. Without technical progress, the condition for reswitching between the two techniques is that equation

$$(15) \qquad \sum_{i=0}^{n} (l_{Ai} - l_{Bi})(1 + r)^{-i} = 0$$

should have more than one positive root. If there is technical progress the condition for reswitching becomes equation

$$(15') \qquad \sum_{i=0}^{n} (l_{Ai} - l_{Bi})\left(1 + \frac{r - h}{1 + h}\right)^{-i} = 0$$

having more than one positive root. The number of switching points remains the same without or with technical progress, but the switching values of r are now different. If without technical progress there is reswitching between two techniques for values of r equal to r_1 and r_2, with technical progress the switching values of r become $[h + (1 + h)r_1]$ and $[h + (1 + h)r_2]$. It might happen that a switching point which without technical progress occurs at positive values of $w(r)$, with technical progress occurs at negative values of $w(r)$ and therefore loses economic significance. On the other hand, it might also happen that a switch point which without technical progress appears at negative values of r and has no economic significance appears now at non-negative interest rates and therefore acquires economic significance. Whenever techniques differ with respect to the sequence of labour inputs, whether or not they differ also with respect to the sequence of their a_i coefficients, there is no reason whatsoever to assume on the ground of realism that technical progress reduces the relevance of the reswitching phenomenon. (The same holds *a fortiori* if technical progress is of the " disembodied " kind, because in that case it does not alter at all the relative profitability of techniques.)

When technical progress occurs, the same relation between w and r holds again between c and g. Let us again call L_t the amount of labour employed on projects currently being started, and define the scale of projects in *to-day's* book of blueprints so that $l_0 = 1$. Let the number of projects started in every period increase, as in the case without technical progress, at a rate g per period. Labour employed on projects started in the previous period, L_{t-1}, is equal to $L_t(1 + h)(1 + g)^{-1}l_1$, and in general labour employed on projects started in the period $t - i$, L_{t-i}, is given by equation (7′):

$$(7') \qquad L_{t-i} = L_t(1 + h)^i(1 + g)^{-i}l_i$$

Gross putty output, X, total material inputs needed to make machines, J, and total consumption, C, are still given by equations (9), (10) and (11), but employment N_t is now given by equation (8′):

$$(8') \qquad N_t = \sum_{i=0}^{n} l_i(1+g)^{-i}(1+h)^i \,.\, L_t$$

which means that the proportion of total employment devoted to starting new projects, L_t/N_t, varies inversely with the rate of technical progress. Consumption per head at time t is accordingly given by equation (13′):

$$(13') \qquad c_t = \frac{\sum_{i=0}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}(1+h)^i}$$

If the rate of growth of employment is equal to that of the labour force, λ, we have now

$$(16) \qquad \lambda = \frac{1+g}{1+h} - 1$$

i.e., $g \simeq \lambda + h$.

It should be noticed that the relation between equations (3″′) and (13′) is the same as that holding between equations (3) and (13), namely $w_t(r) = c_t(g)$ for $r = g$ so that the golden rule is not altered by the presence of technical progress of this kind.[1]

VI. Income and Capital

So far we have discussed the problems of growth and technical choice without having to measure the value of " machines " in terms of consumption goods (except that we have stipulated that the value of an investment option, *i.e.*, of a machine not yet built, must be zero). If we want to measure " income " according to international statistical conventions, however, the relative prices of machines of all ages in terms of consumption goods are needed, as the income produced in one period is a collection of heterogeneous objects, made of whatever happens to be in existence at the end of the period, *minus* whatever was in existence at the beginning of the period, *plus* what has been withdrawn from the productive system in the form of consumption.

Call v_j the value in terms of consumption goods (putty) of a machine used in a given process of a unit scale at the beginning of period j of its existence (or, more generally, the value at time t of having " access to " a unit scale process started at time $t - j$). Suppose there is no technical

[1] If real wages increase at a rate different from $h = \frac{d}{1-d}$, or if technical progress is not neutral in the sense defined above, of course there can be no steady state growth.

progress, wages are paid at the end of the period, and either there is no money or prices are constant in time. The value of a machine is given by

$$v_j = \sum_{i=j}^{n} [a_i - l_i w(r)](1 + r)^{-i}, \quad j = 0, \ldots n \tag{17}$$

The value v of a piece of equipment embodying a given technique depends on its age j and the rate of interest r. We know that $v_0 = 0$ for the technique which is best at any given interest rate, by the very definition of $w(r)$ (see equation (3)). For a given technique, however, the " price " Wicksell effect $\frac{dv_j}{dr}$ and the " ageing " effect $[v_{j+1}(r) - v_j(r)]$ can in principle take either sign. When there are many techniques the level of the interest rate will determine *which* of the techniques is in use as well as the relative value of the different processes at each period of their operation.

From equation (7) we can obtain the number of machines of all ages in existence, so that the value of the capital stock of an economy will be given by

$$V_t = L_t \, . \sum_{j=1}^{n} v_j(1 + g)^{-j} \tag{18}$$

which from (17) can also be written as

$$V_t = L_t \, . \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j} \tag{18$'$}$$

In steady growth net investment I_t undertaken during period t is given by

$$L_t = g \, . \, L_t \sum_{j=1}^{n} v_j(1 + g)^{-j} \tag{19}$$

which can also be written as

$$I_t = g \, . \, L_t \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j} \tag{19$'$}$$

Income produced during period t, $Y_t = C_t + I_t$, from (11) and (19′) can be written as

$$Y_t = \tag{20}$$
$$L_t \left\{ \sum_{i=0}^{n} a_{i,r}(1 + g)^{-i} + g \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j} \right\}$$

Income per head, $y = y(r, g)$, can be obtained from (20) and (8):

$$y = \frac{\sum_{i=0}^{n} a_{i,r}(1 + g)^{-i} + g \, . \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j}}{\sum_{i=0}^{n} l_{i,r}(1 + g)^{-i}} \tag{21}$$

The value of output per man in an economy with access to a given technology depends on the interest rate, which determines the technique chosen (if many are available) and the relative prices of machines and consumption goods, and on the growth rate, which determines the weight of each kind of commodity in output.

If there is only one technique we have that if $g = 0$, $y = c(0)$; if $r = 0$, $y = w(0) = c(0)$, so that we can say that $y(0, g) = y(r, 0)$. If the rate of interest is zero the value of output per man does not vary with the growth rate; if the growth rate is zero the value of output per man does not vary with the interest rate; and the value of output per man is the same in both cases.

If there are many techniques this is not necessarily the case. If $g = 0$, $y = c(r, 0)$; if $r = 0$, $y = w(0) = c(0,0)$. If the interest rate is zero the value of output per head still does not vary with the growth rate; but if the growth rate is zero the value of output per head will vary with the interest rate, and the two will be the same only if r is in the range for which $c(r, 0) = w(0)$.

The value of " capital per man " in the economy is given by (8) and (18′):

$$\frac{V_t}{N_t} = \frac{\sum_{j=1} \sum_{i=j}^{n} [a_{i,r} - l_{i,r}w(r)](1 + r)^{-i}(1 + g)^{-j}}{\sum_{i=0}^{n} l_{i,r}(1 + g)^{-i}} \tag{22}$$

As we saw in Section III, unless one has *faith* that the nature of technology is such that reswitching of techniques does not occur there is no reason to assume that each technique will be associated with a single value or range of values of the interest rate. But even if there is no reswitching, for a given growth rate the same *value* of capital per man can occur at more than a single level or range of the interest rate; or, conversely, for a given interest rate the same value of capital per man can occur at more than a single level or range of the growth rate.[1]

The concept of " value of capital " therefore does not add anything to the analysis of the problems of choice of production techniques for the capitalist firm and the socialist planner. The values associated with a given technique of production *depend on* the criterion and parameters of technical choice, and therefore *cannot provide* themselves any criterion or parameters on which technical choice could be based.

The analysis of the notions of income and capital could be easily extended to the cases where there is technical progress, wages are paid at the beginning of the period and price level is not constant, but the nature of the problem would remain unchanged.

[1] This has been pointed out by L. Spaventa, " Realism without Parables in Capital Theory," in CERUNA, *Recherches récentes sur la fonction de production* (Namur, 1968); *Rate of Growth, Rate of Profit, Value of Capital per Man* (mimeographed); and P. Garegnani, *Heterogeneous Capital, the Production Function and the Theory of Distribution* (mimeographed).

VII. Macroeconomic Equilibrium under Socialism and Capitalism

If we rule out international borrowing and lending the maintenance of equilibrium growth requires that actual consumption per head should be equal to consumable output $c = c\,(r, g)$, whatever the actual relation between r and g. Equilibrium relations must therefore hold between growth rate, interest rate and saving propensities. This, however, poses different problems under socialist and capitalist conditions.

The socialist planner will provide a certain amount of collective consumption per head, $z > 0$; will collect the voluntary savings of workers who will save, say, a fraction s_w of their net wages; will collect a fraction b of workers' wages in taxes, or pay out a corresponding subsidy of $b < 0$. As long as the planner can choose b and z, he can ensure that the condition is satisfied

$$z + (1 - s_w)(1 - b)w(r) = c(r, g) \tag{23}$$

and obtain simultaneously equilibrium growth and the desired balance between private and collective consumption. This is true whether or not he sticks to the "golden rule," whether he chooses the technique himself, or instructs state managers to use the present-value maximisation criterion. As long as equation (14) is satisfied, the excess of current putty output per head over c will be exactly equal to the amount required to maintain the rate of growth g, because this is exactly how we have defined c in equations (11) and (13). The interest rate workers get on their savings is presumably negligible, because the socialist planner does not want them to turn into *rentiers*, but even if they get the full rate r, the planner can always adjust z and b to obtain (14). If $w > c$, out of what is collected by the planner from the workers in the form of savings and taxation, $(s_w + b - bs_w)w$, an amount $(w - c)$ per man employed will have to be lent each period to firms via the credit system. If $c > w$ the planner will use the excess of firms' repayments and interest payments over current loans to firms, equal to $(c - w)$ per man employed, to finance collective consumption or to subsidise wages. From one period to another, if $g \neq 0$ the stock of machines of all ages (in gestation, new, used) will grow (or decline) at a rate g, the machine-mix depending on g, but unless he has to comply with international statistical agreements, the planner does not have to assess the "value" of the State's capital stock and its net change in time (net investment). All he might want to know is the sum of gross output which is due to come in the future from the stock of machines already existing in the economy. Let us call ρ the rate at which he discounts future output (this can be equal to zero, or to the interest rate he charges state firms, or it can take some other value). At the beginning of time t there are $L_t \,.\, (1 + g)^{-j}$ machines of age j in existence.

The cumulative discounted putty-output A_j of a machine of age j is given by

$$(24) \qquad A_j = \sum_{i=j}^{n} a_i(1+\rho)^{-i}$$

Total cumulative gross putty output A_t from the stock of machines already existing in the economy at the beginning of time t is therefore given by

$$(25) \qquad A_t = \sum_{j=0}^{n} \sum_{i=}^{n} a_i(1+\rho)^{-i} \, . \, L_t(1+g)^{-j}$$

He might want to calculate A_t excluding unfinished machines, in which case the sum is taken only for $j = k, \ldots n$. He has no reason to subtract wage costs from future putty output: if, however, he wants a measure of discounted future *surplus* of output over *necessary* labour inputs he will subtract the *subsistence wage* rather than $w(r)$. All these measurements have no interest for the managers of state firms. If they happen to exchange machines and putty with each other they will assess the value of a machine in the same way as a capitalist manager would (*i.e.*, subtracting from future output the expected wage costs as in equation (17)). Their measure, in turn, is of no interest for the planner: if they have followed his instructions of maximising the present value of their assets, in a competitive labour market, the value of their assets assessed from their point of view is equal to their outstanding liabilities to the State. The planner knows this magnitude from his books, but it is a purely accounting notion of no operational significance from *his* point of view.

The planner is " making profits " in the sense that if $g > 0$ production of machines in each period exceeds the replacement of machines which have come to the end of their physical lifetime; if $g < 0$ he is only making a " gross profit." Since profits are only the measure of investment undertaken, and in this sense are " reinvested " by definition, there is no need for measuring profits, *i.e.*, the net change in time of the capital stock. Within the framework outlined in this paper, this is true even in a socialist economy where " profits " are used as a source of bonus payments (to the managers and workers) and investment finance, because if all managers are equally efficient, profits in equilibrium should be maximum and equal to zero. If managers are not homogeneous, and managerial abilities need material rewards to come forward, infra-marginal managers would secure quasi-rents to their firms. At the ruling interest rate they would be able to pay a wage higher than that offered by the marginal manager, but they will actually pay the same rate as he does. Given whatever limits the size of their undertakings, infra-marginal managers will obtain quasi-rents equal to the numbers of workers they employ times the difference between the wage-rate they could afford to pay and the wage-rate offered by the marginal manager. The value of their assets, again, would not have to be assessed to compute their " profits." Even under this form of decentralised socialism, which we could call " managerial socialism " to stress the role of managers in the decision-making process and the enjoyment of profits, the socialist planner could still make sure that actual total consumption does not exceed nor fall short of the level consistent with the maintenance of full-employment growth. In addition to the usual instruments of economic policy (namely, the choice of the level of collective consumption and wage taxation of subsidising), the planner could lay down rules about the share of profits retained by enterprises and the way they should be divided among managers and workers and between consumption and investment.

The problem of macroeconomic equilibrium and the role of profits and capital are, of course, entirely different in a capitalist economy. Whatever is produced in excess of what is needed to pay wages accrues to the capitalists in the form of profits; the evaluation of profits requires the evaluation of machines; capitalists might consume part of their profits; workers will get an interest rate on their savings comparable to that of capitalists. Unless there is state intervention, additional equilibrium relations will have to hold between saving propensities, output and consumption per head, rates of interest and growth. Let us suppose that all investment has to be financed out of profits, *either* because the workers' propensity to save is zero *or* managers of firms have the power to retain part of the profits and distribute the rest to shareholders, and both workers and shareholders have a zero propensity to save (so that s is equal to the retention ratio); *or* workers have a propensity to save $s_w > 0$, but this entitles them to control over a share of total profits equal, in steady state, to their share in current savings.[1] When this is the case we can write the equilibrium condition as

$$(1 - s)[y(r, g) - w(r)] = c(r, g) - w(r) \tag{26}$$

where s is the propensity to save out of profits. Whenever $y > w$, the equilibrium value of s, s^*, corresponding to a given pair of values of r and g is given by

$$s^* = \frac{y(r, g) - c(r, g)}{y(r, g) - w(r)} \tag{27}$$

Suppose a capitalist economy is organised according to the golden rule of accumulation so that $r = g$: in this case $c = w$, and it follows from (27) that the only equilibrium value of the saving propensity of capitalists is

[1] The relation between growth rate, saving propensities, profit rate and distributive shares has been put forward by N. Kaldor, " Alternative Theories of Distribution," *Review of Economic Studies*, 1956; J. Robinson, *The Accumulation of Capital* (1956); and generalised by L. L. Pasinetti, " Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth," *Review of Economic Studies*, 1962. Pasinetti has shown that if workers receive an interest payment on their savings equal to that of capitalists, under certain conditions the propensity to save of workers does not affect the determination of the profit rate and the distributive shares. This proposition has been further discussed by P. A. Samuelson and F. Modigliani, N. Kaldor, J. Robinson and L. L. Pasinetti in *The Review of Economic Studies*, 1966.

unity. It follows that capitalist exploitation takes two forms: one is the capitalists' acquisition of consumption of goods through straightforward command over other people's labour; the other, more subtle form of exploitation is the lower average level of consumption per head associated with a suboptimal technical choice, whenever consumption out of profit prevents the fulfilment of the golden rule. (It should be emphasised again, perhaps, that the golden rule yields optimal technical choice only in conditions of steady state growth, if the criterion of optimality is taken to be the highest rate of steadily growing consumption per head; out of steady state or with a different optimality criterion the rule would not necessarily hold.)

Whenever the saving propensity of capitalists is less than unity, for each steady growth rate there will be one, or possibly many pairs of values of r and s^*. Given the constraint $1 \geqslant s \geqslant 0$, if w is a decreasing function of r we have $c(r, g) < w(r)$ for all $r < g$: for the constraint to be satisfied the growth rate must not exceed the interest rate.

In a capitalist as in a socialist economy, the notion of " value of capital " is not necessary to determine technical choice. In a planned socialist economy the only relevant parameters are the consumption per head—and its behaviour in time if there is technical change or the economy is out of a steady state—and the growth rate of employment. The concept of " value of capital," however, is indispensable to the political economy of capitalism because it performs two fundamental roles, one practical and one ideological.

At a practical level the evaluation of machines of different kinds and different ages in terms of output is needed to settle transactions among capitalist firms, to determine the value of the legal exclusive right to use machinery, and the value of the pieces of paper embodying such rights. It is necessary to determine distribution of income not between the haves and the have-nots but among the haves.

The ideological role of " the value of capital " is that of breaking the direct actual link between the *time pattern* of labour inputs and the *time pattern* of output in which any technology can be resolved, and establishing instead a relation between *current* output and *current* labour. To this purpose the *current* " value of the capital stock " is needed; a mythical conceptual construction in which the past and the future of the economy are telescoped into the present. Attention is focused not on past labour but on the present value of the embodiment of past labour, and its current productiveness can be taken to provide a justification for the attribution of the surplus of current output over the wage bill to those who have appropriated the embodiment of past labour, thereby providing the current basis of future appropriation.

Domenico Mario Nuti

King's College,
Cambridge.

Mathematical Appendix

Equations (3) and (13) have the form

$$f(x) = \frac{a(x)}{l(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} l_i x^i}$$

The equations differ in that in (3) $x = \dfrac{1}{1+r}$, $r \geqslant 0$, so that x lies in (0, 1), while in (13) $x = \dfrac{1}{1+g}$, $g > -1$ and x lies in $(0, \infty)$. We shall analyse[1] $f(x)$ under the following conditions, common to both (3) and (13):

I: x lies in $(0, \infty)$

II: $l_0 = 1, l_n > 0, l_i \geqslant 0$ for $i = 1, \ldots (n-1)$

III: $a_i \leqslant 0$ for $i = 0, \ldots (k-1)$, $a_k > 0$, $a_n > 0$

and

$a_i \geqslant 0$ for $i = (k+1), \ldots (n-1)$; $k \geqslant 1$

IV: $\sum_{i=0}^{n} a_i > 0$

If no a_i is negative, $a(x) > 0$ for all $x > 0$ and $a(0) = 0$.

Suppose q is the largest i such that $a_i < 0$. Then if $a'^{(p)}$ is the pth derivative of a, and $p \leqslant q$, $a'^{(p)}(x) < 0$ for small x and $a'^{(p)} \to \infty$ as $x \to \infty$, together with Descartes' rule of signs, show that $a'^{(p)}$ has one, and only one, zero in $x > 0$. Also its turning-point (*i.e.*, the solution of $a'^{(p+1)} = 0$), if it exists, must occur at smaller x than its zero (the solution of $a'^{(p)} = 0$). For $p > q$, $a'^{(p)}$ has no zero or turning-point.

Similarly, for all p, $l'^{(p)}$ has no zero or turning-point in $x > 0$, and $l'^{(p)} \to \infty$ as $x \to \infty$, except for $l'^{(n)} = n\,!\,l_n$. $l'^{(p)} > 0$ for $x > 0$.

Now consider $g_p = \dfrac{a'^{(p)}}{l'^{(p)}}$ (Note $f = g_0$). This is defined and finite for all $x > 0$. At $x = 0$, $g_p = \dfrac{a_p}{l_p}$, or if $l_p = 0$ but $l_m \neq 0$ (m being the least number greater than p which satisfies this condition), then as $x \to 0$, g_p is approximately proportional to $x^{-(p+1-m)}$. As $x \to \infty$, $g_p \to \dfrac{a_n}{l_n}$.

$g_p = 0$ if, and only if, $a'^{(p)} = 0$, so g_p has one, and only one, zero, for $p \leqslant q$, and the zero of g_{p+1} occurs at smaller x than that of g_p.

$g'_p = 0$ if and only if $\dfrac{(g_{p+1} - g_p)l'^{(p+1)}}{l'^{(p)}} = 0$, *i.e.*, if, and only if, $g_p = g_{p+1}$,[2] and g'_p has the same sign as $g_{p+1} - g_p$. Thus g_p cannot cross g_{p+1} from below (above) when g_{p+1} is increasing (decreasing). If g_{p+1} has a maximum or mini-

[1] I am greatly indebted to Malcolm MacCallum, who provided this analysis, including the result on the maximum number of turning-points of $f(x)$ and its proof.

[2] For the case $p = 0$, this was pointed out to us by the Hon. C. Taylor.

mum, and g_p were to meet it there and hence have a maximum or minimum, this would violate the condition that g'_p has the same sign as $g_{p+1} - g_p$, since one would change sign and the other not. For the same reason if g_p were to have a point of inflexion at the crossing of g_{p+1}, then g_{p+1} must also have one, and by repetition so must g_{n-1} and g_n. But g_n is constant, and so g_{n-1} is either constant or monotone. Thus the only exceptional case is where all g_p are constant, which is ruled out by III.

Thus we see that between any two turning-points of g_p there must be a turning-point of g_{p+1}, so g_p has at most one more turning-point than g_{p+1} (if this were not so, the condition that g'_p and $g_{p+1} - g_p$ are of the same sign is violated).

For this to happen we must have g_p initially increasing if g_{p+1} is initially decreasing, and vice-versa. This is to say that $g_{p+2}(0) - g_{p+1}(0)$ and $g_{p+1}(0) - g_p(0)$ must be of opposite sign. Note that $g_p(0) = a_p/l_p$. There are two exceptional cases, one when $l_p = 0$ and one when $g_{p+1}(0) = g_p(0)$.

A. If $g_{p+1}(0) = g_p(0)$, then $g'_p(0) = 0$ and $g'_{p+1}(0)$ has the same sign as $g_{p+2}(0) - g_{p+1}(0)$. Hence $g_p \lesseqgtr g_{p+1}$ for sufficiently small x according as $g_{p+1}(0) \lesseqgtr g_{p+2}(0)$. By repetition of this argument we see that zeros in the sequence are to be ignored.

B. If $l_p = 0$, $g_p \to \pm\infty$ as $x \to 0$, and so we count $g_{p+1}(0) - g_p(0)$ as positive if a_p is negative, and negative if a_p is positive. Since when $l_r = 0$ $(r = h \; . \; . \; . \; p)$ and $l_{p+1} \neq 0$, we have $g_p \approx x^{-(p-h)}$ for small x, we must count $g_{r+1}(0) - g_r(0)$ as negative if a_r positive, and positive if a_r negative.

Theorem 1. The number of turning-points of $f(x)$ under conditions I–IV has a maximum s, s being the number of alternations of sign of $g_p(0) - g_{p-1}(0)$ as p decreases from n to 1, exceptional cases being covered by A and B above.

The proof is above. The extension to the case $l_n = 0$ is easy.

We know g_k has at most $(m - 1)$ turning-points, where $(m - 1)$ is the number of alternations of sign of $g_p(0) - g_{p-1}(0)$ in $p = n, \; . \; . \; . \; (k + 1)$. If $g_{k+1}(0) < g_k(0)$, g_q can have at most m turning-points, all being at positive values of g_q. Since g_{q-1} has its zero at a larger x than g_q, g_{q-1} has at most m turning-points at positive g_{q-1}, and repeating we have:

Theorem 2. The number of turning-points of $f(x)$ under conditions I–IV above which occur at positive values of $f(x)$ is m, where m is the number of alternations of sign of $g_{p+1}(0) - g_p(0)$ (using rules A and B) in $p = n, \; . \; . \; . \; q$.

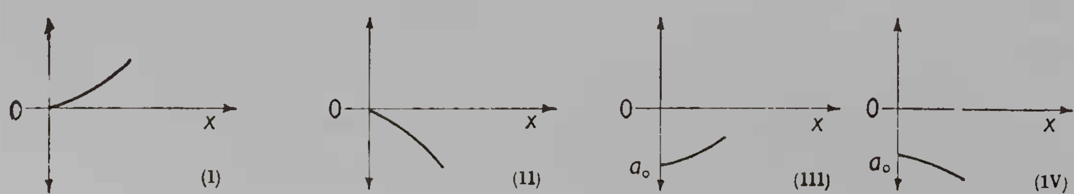
Corollary. The number of turning-points of $f(x)$ at negative values of $f(x)$ is at most $(s - m)$.

Examples. 1. If a_p/l_p increases steadily for $p = 1, \; . \; . \; . \; n$, $f(x)$ has no turning-points at positive $f(x)$.

2. If a_p/l_p increases steadily for $p = q, \; . \; . \; . \; v$ and decreases for $p = v, \; . \; . \; . \; n$ $(q < v < n)$, $f(x)$ has one turning-point at positive $f(x)$.

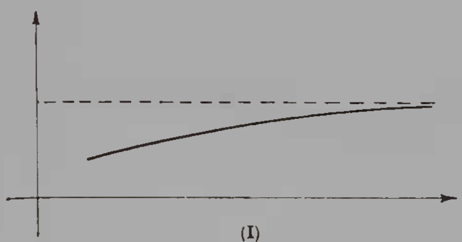
Thus the properties of $f(x)$ are as follows:

1. $f(x)$ starts in one of four ways:

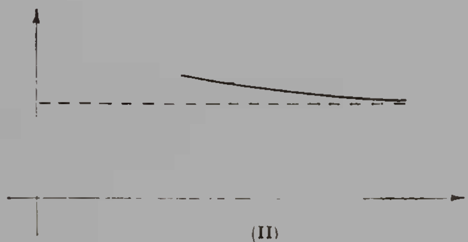


2. In cases II–IV it has one, and only one, zero, in (0,1) as $\sum_{i=0}^{n} a_i > 0$ and $f(1) = \sum_{i=0}^{n} a_i$. In case I it has no zero.

3. It has two ways of ending:



(I)



(II)

4. The number of turning-points of $f(x)$ has a maximum given by the theorems above.

Once we know the properties of the function $f(x)$, we can infer the properties of f as a function of r, say $z(r)$, and/or g, which are the actual variables we want economically. We note that $r = \frac{1}{x} - 1$ or $g = \frac{1}{x} - 1$ as appropriate.

$$z(0) = f(1) > 0$$
$$z_{r \to \infty} = f(0)$$
$$z(-1) = f_{x \to \infty}$$

If x^* is a zero of $f(x)$, $z\left(\frac{1}{x^*} - 1\right) = 0$.

The number of turning-points of $z(r)$ for r in $(-1, \infty)$ or $(0, \infty)$ is the same as the number of turning-points of $f(x)$ for x in $(0, \infty)$ or $(0, 1)$ respectively.

METROECONOMICA

Vol. XXV Gennaio-Aprile 1973 Fasc. I

THE NOTION OF VERTICAL INTEGRATION IN ECONOMIC ANALYSIS (*)

by Luigi L. Pasinetti

Very few notions in economic analysis are so seldom explicitly mentioned as the notion of vertical integration and are at the same time so widely used, implicitly or without full awareness (1). I came to this conviction during the discussions on a multi-sector model of economic growth which I presented a few years ago (2). The synthetic

(*) This paper was presented for discussion at the « Gruppo per lo studio dei problemi della distribuzione del progresso tecnico e dello sviluppo economico » C.N.R. Rome (Italian National Research Council), on December 13, 1972. I am grateful for useful comments to Piercarlo Nicola, Paolo Varri, Sergio Parrinello, Antonio Gay.

(1) The notion of vertical integration is implicit in all discussions on the theory of value of the Classical economists. The same thing can be said of the marginalist economists. When, for example, Léon Walras adopted the device of eliminating intermediate commodities from his analysis of production, he was making use of the logical process of vertical integration. (See « Elements of Pure Economics », W. Jaffé, ed., pp. 241 and ff.). Keynesian macro-economic analysis is also generally carried out in terms of vertically integrated magnitudes (net national income, net savings, new investments consumption, etc.). Very rarely, however, is the logical process of vertical integration explicitly discussed. Generally it is simply taken for granted.

(2) *A New Theoretical Approach to the Problems of Economic Growth*, in « The Econometric Approach to Development Planning », « Pontificiae Academiae Scientiarum Scripta Varia », no. 28, Vatican City 1965, (republished by North Holland Publishing Co., Amsterdam 1965), pp. 571-696; to be referred to, in the following pages, simply as *New Theoretical Approach*.

— 2 —

notion of a « vertically integrated sector » is used explicitly in that model, but within the simplified context of an economic system in which capital goods are made by labour alone; and I have always been faced with questions (3).

An explicit and more general investigation of the meaning and relevance of vertical integration in economic analysis may therefore prove of some usefulness. Instead of starting from the synthetic notions and going back to their elementary components, I shall start here from these elementary components — i.e., from the now familiar schemes of interindustry analysis — and go on to the synthetic notions. The crucial role played by vertical integration in the theories of value, income distribution and economic growth should emerge clearly as the investigation develops. The whole analysis will be carried out with reference to the general case of production of all commodities by means of commodities. The simplified case of capital goods produced by labour alone will be shown at the end as a particular case.

1. Production of commodities by means of commodities

An economic system will be considered in which all commodities are produced by means of commodities, used as capital goods. Commodities enter the process of production at the beginning of each « year » as inputs, jointly with labour services, and commodities come out at the end of the year as outputs. The economic system is supposed to be *viable*, in the sense that it is capable of producing larger quantities of commodities than those required to replace used-up capital goods.

The following notation will be used throughout (4):

i) column vector $\mathbf{X}(t) \equiv [X_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical quantities of the m commodities that are produced in year t;

ii) column vector $\mathbf{Y}(t) \equiv [Y_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical net product of the economic system, i.e. what is available for consumption and new investments after deducting replacements from $\mathbf{X}(t)$. Of course $\mathbf{Y}(t)$ may further be regarded as a sum of commodities devoted to consumption and of commodities devoted to investments, to be denoted by column vectors $\mathbf{C}(t) \equiv [C_i(t)]$ and $\mathbf{J}^{(d)}(t) \equiv [J_i^{(d)}(t)]$, $i = 1, 2, \ldots m$, respectively. By definition, $\mathbf{C}(t) + \mathbf{J}^{(d)}(t) \equiv \mathbf{Y}(t)$;

---

(3) These questions have normally been concerned with the problem of how to construct the vertically integrated sectors in the general case. Some indications are given in Chapter VI of *New Theoretical Approach*, but in a brief and incomplete way.

(4) Letters in bold type will be used to denote vectors and matrices. Symbol $\equiv$ will be used to denote a definitional equality and in particular, as in all cases in this section, an equality of two different notations for the same thing.

— 3 —

iii) column vector $\mathbf{S}(t) \equiv [S_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical quantities of commodities that are required as capital goods (capital stocks), at the beginning of year t, in order to obtain quantities $\mathbf{X}(t)$ at the end of year t;

iv) row vector $\mathbf{p}(t) \equiv [p_i(t)]$, $i = 1, 2, \ldots m$, to denote prices of commodities $1, 2, \ldots m$;

v) scalar $L(t)$, to denote the labour force required by the economic system in year t, measured, let us say, in man-years;

vi) scalar π to denote the (uniform) rate of profit;

vii) scalar $w(t)$ to denote the (uniform) wage rate.

As far as technology is concerned, two successive analytical stages will be taken — a procedure which is by now customary in this type of analysis. Usually, as is well known, one begins by considering, first, production with circulating capital goods; then one goes on to production with fixed capital goods. The advantage of this procedure is that almost all basic concepts can be singled out at the first stage, where relatively few analytical complications arise. The second stage can then be devoted to pointing out which conclusions still hold and which conclusions are affected by generalization. A slightly more general approach is taken in the present work. Fixed capital goods are introduced immediately at the first stage, but with a simplifying assumption on how they depreciate. The case of production with fixed capital goods in general will, of course, be considered at a second stage.

2. Fixed capital goods with a simplifying assumption

We shall begin by considering a technology which requires both circulating capital goods (which are used up within one year) and fixed capital goods (which last for more than one year). The simplifying assumption will be made that in each industry j a constant proportion δ_j of all fixed capital goods drops out of the production process each year $(j = 1, 2, \ldots m)$. Moreover, for the time being, all technical coefficients will be supposed to be constant through time.

The technique of the whole economic system will be represented by:

i) a row vector $\mathbf{a}_{[n]} \equiv [a_{nj}]$, $n = m + 1$, $j = \;, 2, \ldots m$, all $a_{nj} \geq 0$, where each a_{nj} denotes the annual input of labour required by one physical unit of the commodity produced in industry j;

ii) a square matrix $\mathbf{A} \equiv [a_{ij}]$; $i, j = 1, 2, \ldots m$, all $a_{ij} \geq 0$, in which each column j represents the physical stocks of capital goods (both circulating and fixed) required for the production of one physical unit of the commodity produced in industry j. Square matrix $\mathbf{A}$ may be regarded as the sum of two non-negative square matrices:

— 4 —

$\mathbf{A}^{(C)} \equiv [a_{ij}^{(C)}]$ and $\mathbf{A}^{(F)} \equiv [a_{ij}^{(F)}]$; $i, j = 1, 2, \dots m$, as representing the stocks of circulating capital goods and the stocks of fixed capital goods respectively. Therefore $\mathbf{A} \equiv \mathbf{A}^{(C)} + \mathbf{A}^{(F)}$, by definition. Of course, each year in each industry j, the economic system has to replace all circulating capital goods and a fraction δ_j of the fixed capital goods; $j = 1, 2, \dots m$. If we call $\hat{\delta}$ a diagonal matrix with alla the δ_j's on the main diagonal, we may therefore define another square matrix $\mathbf{A}^{\ominus} \equiv [a_{ij}^{\ominus}] \equiv \mathbf{A}^{(C)} + \mathbf{A}^{(F)}\hat{\delta}$, as representing that part of the initial stocks of capital goods that are actually used up each year by the production process. By definition, $\mathbf{A}^{\ominus} + \mathbf{A}^{(F)}(\mathbf{I} - \hat{\delta}) \equiv \mathbf{A}$. The particular case in which all capital goods are circulating capital goods is represented by $\mathbf{A}^{(F)} = \mathbf{O}$ and, therefore, $\mathbf{A}^{\ominus} = \mathbf{A}$.

With this notation, the physical economic system may be represented by the following system of equations:

$$(\mathbf{I} - \mathbf{A}^{\ominus})\, \mathbf{X}(t) = \mathbf{Y}(t)\,, \tag{2.1}$$

$$\mathbf{a}_{[n]}\, \mathbf{X}(t) = \mathrm{L}(t)\,, \tag{2.2}$$

$$\mathbf{A}\, \mathbf{X}(t) = \mathbf{S}(t)\,, \tag{2.3}$$

where (2.1), (2.2) represent the flows of commodities and labour services required in year t to produce net product $\mathbf{Y}(t)$, and (2.3) represents the stocks of capital goods required at the beginning of year t for production to be effected. At the same time, equilibrium prices are represented by the following system of equations:

$$\mathbf{p} = \mathbf{a}_{[n]}\, w + \mathbf{p}\mathbf{A}^{\ominus} + \mathbf{p}\mathbf{A}\pi, \tag{2.4}$$

which determines all prices if one of these and the wage rate (or alternatively the rate of profit) are fixed exogenously.

3. An « industry »

On the assumption just made concerning fixed capital goods, each industry j ($j = 1, 2, \dots m$) produces only one good: commodity j; and in order to produce one physical unit of such a commodity, it needs a quantity of labour represented by the j^{th} coefficient of vector $\mathbf{a}_{[n]}$ and a series of heterogeneous stocks of capital goods, represented by the j^{th} column of matrix $\mathbf{A}$. Industry j may therefore be synthetically represented by a « direct labour coefficient » — the j^{th} component of vector $\mathbf{a}_{[n]}$ — and by what may be called a « unit of direct productive capacity » — a composite commodity defined by the j^{th} column of matrix $\mathbf{A}$.

In equations (2.1) — (2.3), the physical quantities of an economic system are classified precisely in this way, i.e., according to the criterion of the « industry ». This classification has the advantage of

— 5 —

being immediately observable; but it maintains our attention at a rather superficial level. A re-classification of the same physical quantities may be obtained on the basis of a conceptually more complex, but analytically far more powerful criterion, which we are now going to consider.

4. A « VERTICALLY INTEGRATED SECTOR »

We may define a new vector $\mathbf{Y}_i(t)$ as a column vector the components of which are all zeros except the i^{th} one, defined here as $Y_i(t)$ — i.e., the i^{th} component of vector $\mathbf{Y}(t)$. Moreover we shall use: scalar $L^{(i)}(t)$ to denote the quantity of labour required, column vector $\mathbf{X}^{(i)}(t)$ to denote the physical quantities of commodities to be produced, and column vector $\mathbf{S}^{(i)}(t)$ to denote the stocks of capital goods required, in the whole economic system, in order to obtain physical quantity $Y_i(t)$ of final good i $(i = 1, 2, \dots m)$.

For each particular net product $Y_i(t)$, we obtain from (2.1) — (2.3):

$$(4.1) \qquad \mathbf{X}^{(i)}(t) = (\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t)\,,$$

$$(4.2) \qquad L^{(i)}(t) = \mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t)\,,$$

$$(4.3) \qquad \mathbf{S}^{(i)}(t) = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t)\,, \qquad i = 1, 2, \dots m\,,$$

.e., in fact, m sub-systems (as Piero Sraffa has called them) [5]. From (4.1) — (4.3) and the definition of $\mathbf{Y}_i(t)$ it follows that

$$(4.4) \qquad \sum_i^m \mathbf{Y}_i(t) = \mathbf{Y}(t) \qquad ; \qquad \sum_i^m \mathbf{X}^{(i)}(t) = \mathbf{X}(t)\,;$$

$$(4.5) \qquad \sum_i^m L^{(i)}(t) = L(t) \qquad ; \qquad \sum_i^m \mathbf{S}^{(i)}(t) = \mathbf{S}(t)\,.$$

The m sub-systems add up to the original complete economic system.

The economic meaning of the coefficients appearing on the right hand side of (4.1) has been widely illustrated in the economic literature. Matrix $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$ is known as the Leontief inverse matrix [6] — its i^{th} column $(i = 1, 2, \dots m)$ contains the series of heterogeneous commodities that are directly and indirectly required in the whole economic system to obtain one physical unit of commodity i as a final good. On the other hand less attention has been paid to the economic meaning of the coefficients that appear on the right hand side of (4.2) and (4.3) [7]. More synthetically, we may define

---

(5) Piero Sraffa, « Production of Commodities by means of Commodities », Cambridge 1960, p. 89.

(6) After Wassily W. Leontief's work « The Structure of American Economy », New York 1941, and 1951.

(7) It is again Wassily W. Leontief who first applied these concepts, in a well known empirical investigation: *Domestic Production and Foreign Trade:*

— 6 —

$$(4.6)\qquad \mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{v} \equiv [v_i]\,,$$

$$(4.7)\qquad \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{H} \equiv [\mathbf{h}_i]\,, \qquad i = 1, 2, \dots m\,,$$

where each $\mathbf{h}_i$ is a column vector, and thus re-write (4.2), (4.3) in a more compact way:

$$(4.2\ \text{b})\qquad L^{(i)}(t) = \mathbf{v}\ \mathbf{Y}_i(t) \equiv v_i Y_i\,,$$

$$(4.3\ \text{b})\qquad \mathbf{S}^{(i)}(t) = \mathbf{H}\ \mathbf{Y}_i(t) \equiv \mathbf{h}_i Y_i\,, \qquad i = 1, 2, \dots m\,.$$

Each coefficient v_i in (4.2b) expresses in a consolidated way the quantity of labour directly and indirectly required in the whole economic system to obtain one physical unit of commodity i as a final good. We shall call it the *vertically integrated labour coefficient* for commodity i ($i = 1, 2, \dots m$). Likewise, each column vector $\mathbf{h}_i$ in (4.3 b) expresses in a consolidated way the series of heterogeneous physical quantities of commodities $1, 2, \dots m$, which are directly and indirectly required as stocks, in the whole economic system, in order to obtain one physical unit of commodity i as a final good ($i = 1, 2, \dots m$). This is another particular composite commodity which we shall call *a unit of vertically integrated productive capacity* for commodity i ($i = 1, 2, \dots m$).

Scalar v_i and column vector $\mathbf{h}_i$, together, represent what we may call the *vertically integrated sector* for the production of commodity i as a final good (whether for consumption of for investment); $i = 1, 2, \dots m$. A vertically integrated sector is therefore a compact way of representing a sub-system, as it synthesizes each sub-system into a single labour coefficient v_i and a single composite commodity $\mathbf{h}_i$. For an economic system with m commodities, we obviously obtain m labour coefficients (the m components of row vector $\mathbf{v}$) and m units of productive capacity (the m columns of matrix $\mathbf{H}$), i.e., m vertically integrated sectors for the production of the m commodities as final goods.

In comparison with the previous section, we may say that vector $\mathbf{a}_{[n]}$ and matrix $\mathbf{A}$ classify the total quantity of labour $L(t)$ and total quantities of stocks of capital goods $\mathbf{S}(t)$ according to the criterion of the industry in which they are required:

$$(4.8)\qquad L(t) = \mathbf{a}_{[n]}\ \mathbf{X}(t)\,; \qquad \mathbf{S}(t) = \mathbf{A}\ \mathbf{X}(t)\,.$$

All these quantities are directly observable and directly quantifiable. Vector $\mathbf{v}$ and matrix $\mathbf{H}$ reclassify the same physical quantities according to the criterion of the vertically integrated sector for which they are directly and indirectly required:

The American Capital Position Re-examined, « Proceedings of the American Philosophical Society », vol. 97, no. 4, Sept. 1953, pp. 332-49; and *Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis*, « The Review of Economics and Statistics », Nov. 1956, pp. 386-407.

— 7 —

$$L^{(i)}(t) = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{Y}_i(t) = \mathbf{v}\,\mathbf{Y}_i(t)\,, \tag{4.2 b}$$

$$\mathbf{S}^{(i)}(t) = \mathbf{A}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{Y}_i(t) = \mathbf{H}\,\mathbf{Y}_i(t)\,, \tag{4.3 b}$$

$$i = 1, 2, \ldots m\,,$$

$$L(\mathrm{t}) = \sum_i^m L^{(i)}(t) \quad ; \quad \mathbf{S}(t) = \sum_i^m \mathbf{S}^{(i)}(t)\,. \tag{4.9}$$

Neither $\mathbf{v}$ nor $\mathbf{H}$ are directly observable, but they can be obtained through post-multiplication by $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$ from quantities $\mathbf{a}_{[n]}$ and $\mathbf{A}$, that are directly observable. They are therefore quantifiable in an indirect way.

To conclude, precisely the same physical quantities $L(t)$ and $\mathbf{S}(t)$ appear in both (4.8) and (4.9), but are classified according to two different criteria — the more immediate criterion of the « industry » in the former, the conceptually more complex criterion of the « vertically integrated sector » in the latter. Both classifications are empirically quantifiable — the former directly and the latter through the indirect logical process of vertical integration.

5. Vertical integration in the theory of value and income distribution

When Adam Smith put forward the proposition that every commodity finally resolves itself into wages, profits and rents ([8]), he rightly sensed that he had reached an important conclusion — he had (implicitly) grasped the basic concept of vertical integration.

In our analysis, neither the price system written in section 2 on the basis of directly observable magnitudes

$$\mathbf{p} = \mathbf{a}_{[n]}w + \mathbf{p}\,\mathbf{A}\Theta + \mathbf{p}\,\mathbf{A}\pi\,, \tag{2.4}$$

nor what may be called its « solution »

$$\mathbf{p} = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta - \mathbf{A}\pi)^{-1}w \tag{5.1}$$

can give us a clear idea of Adam Smith's theoretical insight.

But if we perform a few logical operations and re-write (2.4) as

$$\mathbf{p}(\mathbf{I} - \mathbf{A}\Theta) = \mathbf{a}_{[n]}w + \mathbf{p}\,\mathbf{A}\pi\,, \tag{5.2}$$

$$\mathbf{p} = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta)^{-1}w + \mathbf{p}\,\mathbf{A}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\pi\,, \tag{5.3}$$

we can see the two notions characterizing a vertically integrated sector reappear. After substitution from definitions (4.6), (4.7), the (5.3) may be written

$$\mathbf{p} = \mathbf{v}\,w + \mathbf{p}\,\mathbf{H}\pi\,. \tag{5.4}$$

([8]) Adam Smith, « The Wealth of Nations », E. Cannan edition, pp. 49 and ff., especially p. 52.

— 8 —

This is a remarkable expression, as it explicitly shows that each price is ultimately made up of only two components: wages and profits [9]. It is precisely the logical operation of vertical integration that makes this evident by consolidating all the complex intermediate stages into one single labour coefficient and one single unit of productive capacity — the former being multiplied by the wage rate and the latter (after being evaluated at current prices) by the rate of profit. It may be noticed that vector **pH** is nothing but a vector of the *m* vertically integrated capital-output ratios multiplied by the price of the final commodity to which they refer. Hence, one alternative way of representing the *m* vertically integrated sectors might be that of using vector **v** (the *m* vertically integrated labour coefficients) and vector **pH** (the *m* vertically integrated capital-per-unit-of-output ratios) [10].

Another property of (5.4) is that it exposes the antagonism of wages and profits in income distribution. When $\pi = 0$, the second addendum vanishes and prices become

$$\mathbf{p} = \mathbf{v}\, w\,. \tag{5.5}$$

Wages are obviously at their maximum, as they absorb the whole purchasing power deriving from prices.

Conversely, when $w = 0$, profits are at their maximum and (5.4) becomes a linear and homogeneous system of equations

$$\mathbf{p}\,(\mathbf{I} - \Pi\,\mathbf{H}) = \mathbf{0}\,, \tag{5.6}$$

where Π stands for the rate of profit corresponding to $w = 0$, or maximum rate of profit. Since the economic system is viable *ex-hypothesis*, Π must be positive. Maximum rate of profit Π also emerges, from (5.6), as the reciprocal of the eigenvalue — which we may call λ — of matrix **H**. Non-trivial solutions require, of course,

$$\det\,(\lambda\,\mathbf{I} - \mathbf{H}) = 0\,, \tag{5.7}$$

an algebraic equation that yields *m* roots for λ. However, since **H** is a non-negative matrix, [11] we know on the basis of the Perron-Frobenius theorem [12] that its maximum eigenvalue λ_{max}: *a*) is a real and positive number; *b*) has a non-negative eigenvector (i.e. non-negative prices) associated with it; *c*) is also the eigenvalue which is

---

[9] No rents are considered in the present scheme.

[10] This is in fact the way suggested in Chapter VI of *New Theoretical Approach*, where the procedure to obtain them is also given (multiplication of direct labour coefficients and direct capital-output ratios by the inverse Leontief matrix).

[11] Both $\mathbf{A}^{\ominus}$ and **A** are non-negative *ex-hypothesis*, and moreover, the technique is supposed to be viable, which implies that $(\mathbf{I}-\mathbf{A}^{\ominus})^{-1}$ is non-negative. It follows that **H** is also non-negative.

[12] See, for example, F. R. Gantmacher, « The Theory of Matrices », New York 1959.

— 9 —

maximum in modulus (i.e. $\lambda_{max} = |\lambda|_{max}$). This is the only root of (5.7) that is economically relevant ([13]) and we shall therefore define straightaway:

$$\Pi = \frac{1}{\lambda_{max}} . \tag{5.8}$$

For any positive π lower than Π, we also know on the basis of the Perron-Frobenius theorem that $(\mathbf{I} - \pi\,\mathbf{H})^{-1}$ is non-negative, so that the general solution of (5.4),

$$\mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,w\,, \tag{5.9}$$

yields non-negative prices and an inverse monotonic relation between π and w, whatever the standard in terms of which w is measured ([14]).

The same problems may be looked at in a more « classical » way if the wage rate itself is used as the *numeraire* of the price system, i.e., if we put $w = 1$. In this case all prices come to be expressed in terms of the wage rate, i.e., in terms of « labour commanded ». But the components of $\mathbf{v}$, the vertically integrated labour coefficients, express what classical economists called « labour embodied » (and Marx simply called « values »). Therefore, wages — by being distributed in proportion to « labour embodied » as appears from (5.4) — can « command » only part of the purchasing power deriving from prices. The difference

$$\mathbf{p} - \mathbf{v} = \mathbf{p}\,\mathbf{H}\,\pi \tag{5.10}$$

is absorbed by profits. « Solution » (5.9) becomes

$$\mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,, \tag{5.11}$$

and may also be regarded as expressing the « transformation » of $\mathbf{v}$ into $\mathbf{p}$, i.e., of Marxian values into prices. The linear operator $(\mathbf{I} - \pi\,\mathbf{H})^{-1}$, where the m units of vertically integrated productive capacity are shown to play a crucial role, represents such « transformation » in logical terms. Only when $\pi = 0$, does « labour commanded » become equal to classical « labour embodied » (and prices to Marxian « values »), i.e.,

$$\mathbf{p} = \mathbf{v}\,, \tag{5.12}$$

while matrix $\mathbf{H}$ drops out of the picture altogether.

([13]) The assumption is made, following Sraffa, that the internal rate of reproduction of non-basic commodities (if there are any) is higher than the internal rate of reproduction of basic commodities.

([14]) Since no price can become negative in terms of *any* standard (within the interval, $0 \leq \pi \leq \Pi$), no price can fall faster than w as π is increased. It follows that π and w (in terms of *any* standard) must be inversely and monotonically related to each other. See the detailed proof given by Piero Sraffa, *op. cit.*, pp. 39-40.

— 10 —

6. A particular unit of measurement for capital goods

We may go back to the physical quantity system. So far in this analysis all commodities have been measured in terms of the physical units that are commonly used to measure them (e.g., tons, bushels, numbers, etc.). But expressions (4.7), (4.3 b) suggest the possibility of an alternative physical unit of measurement for capital goods. More precisely they suggest the possibility of measuring capital goods in terms of a particular composite commodity which we may call « physical unit of vertically integrated productive capacity ».

There clearly exists one such physical unit for each final good that is produced. If there are m final goods, there exist m physical units of vertically integrated productive capacity, represented by the columns of matrix $\mathbf{H}$, i.e. by,

$$(6.1) \qquad \mathbf{h}_i = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{e}_i\,, \qquad i = 1, 2, \ldots m\,,$$

where $\mathbf{e}_i$ is the i^{th} unit column vector.

For the purpose of our analysis, a composite commodity does not present any conceptual difficulty. (As a matter of fact, any commodity — e.g., a pair of shoes — can always be considered as composed of various elementary commodities — i.e., leather, string, rubber, etc. — put together in fixed proportions). Therefore, when such units are used, the existing stocks of capital goods may be represented by an m component column vector,

$$(6.2) \qquad \mathbf{K}\,(t) \equiv [K_i\,(t)]\,, \qquad i = 1, 2, \ldots m\,.$$

It follows from definition that, in equilibrium,

$$(6.3) \qquad K_i\,(t) = Y_i\,(t)\,, \qquad i = 1, 2, \ldots m\,.$$

It is always possible to « translate » capital goods expressed in terms of vertically integrated productive capacities into capital goods expressed in ordinary physical units by the transformation

$$(6.4) \qquad \mathbf{S}\,(t) = \mathbf{H}\;\mathbf{K}\,(t)\,.$$

Matrix $\mathbf{H}$ thereby appears as a linear operator which — when applied to a vector of physical quantities measured in terms of vertically integrated productive capacities — reclassifies them in terms of ordinary physical units. When $\mathbf{H}$ is a non-singular matrix, there even exists a unique inverse transformation

$$(6.5) \qquad \mathbf{K}\,(t) = \mathbf{H}^{-1}\,\mathbf{S}\,(t)\,.$$

But of course $\mathbf{H}^{-1}$ need not necessarily exist. (That is: there may be more than one way, or there may be no exhaustive way, of forming vertically integrated units of productive capacity from arbitrarily given existing stocks of ordinary capital goods).

— II —

7. Vertically integrated sectors for investment goods expressed in physical units of vertically integrated productive capacity

When capital goods are measured in physical units of vertically integrated productive capacity, new investments (which are additions to the existing stocks of capital goods) must be measured in the same units. But new investments are considered to be final goods and we know that it is possible to conceptually construct a vertically integrated sector in correspondence to each final good. Such a logical construction has been obtained in section 4 for final goods measured in ordinary physical units. It now becomes possible to obtain similar logical constructions for investment goods measured in physical units of vertically integrated productive capacity.

We may denote by $\mathbf{J}^{(v)}(t) \equiv [J_i^{(v)}(t)]$, $i = 1, 2, \dots m$, the column vector of new investments measured in units of vertically integrated productive capacity for the corresponding final goods $1, 2, \dots m$. And by $\mathbf{J}_i^{(v)}(t)$, $i = 1, 2, \dots m$, a column vector whose components are all zeros except the i^{th} one which is equal to $J_i^{(v)}(t)$. Obviously, $\sum_i^m \mathbf{J}_i^{(v)}(t) = \mathbf{J}^{(v)}(t)$. It follows from (6.4) that $\mathbf{J}^{(v)}(t)$ — new investments expressed in physical units of vertically integrated productive capacity — and $\mathbf{J}^{(d)}(t)$ — new investments expressed in ordinary (direct) physical units — are related by

$$\mathbf{J}^{(d)}(t) = \mathbf{H}\,\mathbf{J}^{(v)}(t)\,. \tag{7.1}$$

Similarly to what has been done in section 4, we may denote by $L^{(k_i)}(t)$ the labour services and by $\mathbf{X}^{(k_i)}(t)$, $\mathbf{S}^{(k_i)}(t)$, respectively, the column vectors of the physical quantities produced, and of the stocks of capital goods required, in the whole economic system, for the production of final good $J_i^{(v)}(t)$. Of course, $i = 1, 2, \dots m$.

For each physical quantity $J_i^{(v)}(t)$ we may now write the corresponding sub-system:

$$\mathbf{X}^{(k_i)}(t) = (\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t)\,, \tag{7.2}$$

from which, after substitution into (2.2), (2.3), we obtain

$$L^{(k_i)}(t) = \mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t) \equiv \mathbf{v}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t)\,, \tag{7.3}$$

$$\mathbf{S}^{(k_i)}(t) = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t) \equiv \mathbf{H}^2\,\mathbf{J}_i^{(v)}(t)\,, \qquad i = 1, 2, \dots m\,. \tag{7.4}$$

Here again scalar $L^{(k_i)}(t)$ is the quantity of labour and vector $\mathbf{S}^{(k_i)}(t)$ is the series of stocks of capital goods directly and indirectly required in the whole economic system in order to produce quantity $J_i^{(v)}(t)$ of the investment good (measured in units of vertically integrated productive capacity) required for final good i. Therefore vector $\mathbf{vH}$ in (7.3), which we may call $\mathbf{v}_k$, i.e.,

— 12 —

(7.5) $$\mathbf{v}_k \equiv \mathbf{vH} \equiv \mathbf{a}_{[n]} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} ,$$

is a vector of vertically integrated labour coefficients and matrix

(7.6) $$\mathbf{H}^2 \equiv \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{H} \equiv \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} ,$$

in (7.4), is a matrix the columns of which represent units of vertically integrated productive capacity. Vector $\mathbf{v}_k$ and matrix $\mathbf{H}^2$ together represent the m vertically integrated sectors for the m investment goods expressed in units of vertically integrated productive capacity.

As was to be expected, the vertically integrated sectors for investment goods, expressed in physical units of vertically integrated productive capacity, have been obtained through a logical operation of vertical integration performed twice.

8. Prices of investment goods expressed in units of vertically integrated productive capacity

When investment goods are expressed in ordinary physical units, their prices are those found in section 5 (i.e., the prices of the commodities of system (2.1) — (2.4), whether they are used for consumption or for investment). But when investment goods are expressed in physical units of vertically integrated productive capacity, their prices — which we may denote by row vector $\mathbf{p}_k \equiv [p_{k_i}]$, $i = 1 , 2 , \ldots m$ — are a weighted average of the prices $\mathbf{p}$ of their elementary components, namely

(8.1) $$\mathbf{p}_k = \mathbf{pH} .$$

After substitution from (5.4) and (7.5), we obtain

(8.2) $$\mathbf{p}_k = \mathbf{v}_k w + \mathbf{p}_k \mathbf{H} \pi .$$

This is a new price system in which prices, instead of being referred to the m ordinary commodities as in system (5.4), are referred to m composite commodities obtained by reclassifying the m ordinary commodities of system (5.4) by the operation of vertical integration (i.e., by multiplication by $\mathbf{H}$). Of course, the price system (5.4) and the price system (8.2) are equivalent. They yield the same maximum rate of profit. (As may be seen, Π emerges here, as in (5.4), as the reciprocal of the maximum eigenvalue of $\mathbf{H}$). And they yield the same maximum wage rate in terms of any pre-assigned standard. If we put $\pi = 0$ and $w = 1$, the components of $\mathbf{p}_k$ again turn out to be equal to the corresponding vertically integrated labour coefficients ($\mathbf{v}_k$ in this case). For all intermediate cases in which $0 < \pi < \Pi$,

(8.3) $$\mathbf{p}_k = \mathbf{v}_k (\mathbf{I} - \pi \mathbf{H})^{-1} w .$$

— 13 —

which gives for $\mathbf{p}_k$ precisely the same general expression that (5.9) gives for $\mathbf{p}$. All remarks and elaborations made for $\mathbf{p}$ in section 5 could therefore be repeated for $\mathbf{p}_k$ here.

9. Vertically integrated sectors of higher order

After performing the logical operation of vertical integration twice, it is natural to ask oneself whether there is any meaning in performing it a third time. The answer is straightforward. The units of vertically integrated productive capacity for investment goods, expressed in units of vertically integrated productive capacity, are themselves composite commodities. We may therefore conceptually construct the vertically integrated sectors for these newly found composite commodities. Such vertically integrated sectors clearly require a logical process of vertical integration to be performed three times. For analytical convenience, we may call such sectors « vertically integrated sectors of the third order » and, therefore, we may now call vertically integrated sectors of the second order, and vertically integrated sectors of the first order, respectively, the logical constructions obtained in section 7 and in section 4.

After using subscript k to denote the vertically integrated labour coefficients of the second order, we shall for consistency use subscript k^2 to denote the vertically integrated labour coefficients of the third order, to be obtained from the second order vertically integrated labour coefficients through post-multiplication by $\mathbf{H}$, i.e.,

$$\mathbf{v}_{k^2} \equiv \mathbf{v}_k \mathbf{H} \equiv \mathbf{a}_{[n]} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{H}\,\mathbf{H}\,. \tag{9.1}$$

No new notation is needed for the matrix of the vertically integrated productive capacities of the third order, which clearly is $\mathbf{H}^3$.

These definitions now allow us to generalize the logical process of vertical integration to any higher order we may like. We can proceed from the vertically integrated sectors of the third order to the vertically integrated sectors of the fourth order, and from those of the fourth order to those of the fifth order, of the sixth order... , and so on step by step to the vertically integrated sectors of the s^{th} order, where s is any natural number as high as we may choose. Analytically each step in this process to a higher and higher order of vertical integration is simply represented by post-multiplication by matrix $\mathbf{H}$. The m units of vertically integrated productive capacity thus play a crucial role in the whole process.

In other terms, we may characterize the m vertically integrated sectors of the s^{th} order by:

a) a vector

$$\mathbf{v}_{k^{s-1}} \equiv \mathbf{a}_{[n]} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \underbrace{\mathbf{H} \ldots \mathbf{H}}_{(s-1)\ \text{times}} \equiv \mathbf{v}\,\mathbf{H}^{s-1} \equiv \mathbf{v}_{k^{s-2}} \mathbf{H}\,, \tag{9.2}$$

— 14 —

the components of which are the m s^{th} order vertically integrated labour coefficients; and by

b) a matrix

$$(9.3) \qquad \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\underbrace{\mathbf{H}\ldots\mathbf{H}}_{(s-1)\ \text{times}} \equiv \mathbf{H}\,\mathbf{H}^{s-1} \equiv \mathbf{H}^{s},$$

the columns of which represent the m s^{th} order vertically integrated physical units of productive capacity.

Each series of m physical units of the s^{th} order vertically integrated productive capacity has of course (associated with it) its own series of m prices, which for consistency we shall denote by row vector $\mathbf{p}_{k^{s-1}}$. We clearly have

$$(9.4) \quad \mathbf{p}_{k^{s-1}} = \mathbf{p}_{k^{s-2}}\,\mathbf{H} = (\mathbf{v}_{k^{s-2}}\,w + \mathbf{p}_{k^{s-2}}\,\mathbf{H}\,\pi)\,\mathbf{H} = \mathbf{v}_{k^{s-1}}\,w + \mathbf{p}_{k^{s-1}}\,\mathbf{H}\pi,$$

from which we obtain

$$(9.5) \qquad \mathbf{p}_{k^{s-1}} = \mathbf{v}_{k^{s-1}}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,w\,,$$

a remarkable general expression, of which (8.3) and (5.9) may be regarded as particular cases. All the theoretical remarks and elaborations made for prices $\mathbf{p}$ in section 5 could now be referred to prices $\mathbf{p}_{k^{s-1}}$ in general.

10. Higher order vertical integration and reduction of prices to a sum of weighted quantities of labour

The notion of higher order vertical integration may at first appear to be a very highly abstract notion indeed, and one may wonder whether any application of it can be found at all. But let us analyse the price system more deeply.

By using first order vertical integration, we have been able in section 5 to split up each price into its two basic components — wages and profits. When the wage rate itself is used as the *numeraire* — i.e., when w is put equal to unity — (5.4) actually becomes

$$(10.1) \qquad \mathbf{p} = \mathbf{v} + \mathbf{pH}\,\pi\,,$$

which shows the two components of prices in yet another light. The total purchasing power of prices, in terms of « labour commanded », is shown to be equal to « labour embodied » plus a residual absorbed by profits. A solution for $\mathbf{p}$ may of course be obtained immediately:

$$(5.11) \qquad \mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,,$$

as was done already in section 5. But an alternative procedure may also be followed — a procedure of successive approximations,

— 15 —

which is conceptually far more interesting from a theoretical point of view.

Residual $\mathbf{pH}\pi$ contains the same prices that appear on the left hand side of (10.1). It may therefore itself be further split up into two components. After substitution from (8.1) and (8.2) we obtain

$$\mathbf{pH}\pi \equiv \mathbf{p}_k\pi \equiv \mathbf{v}_k\pi + \mathbf{p}_k\,\mathbf{H}\pi^2 \,. \tag{10.2}$$

Second order vertical integration has thereby come on to the scene. The two components of $\mathbf{pH}\pi$ are shown to be: profits on the second order vertically integrated labour coefficients and a second order residual, itself containing $\mathbf{p}_k$. A chain argument has been started. Residual $\mathbf{p}_k\,\mathbf{H}\pi^2$ may itself be split up into two further components by using the notion of third order vertical integration. After substitution from (9.4) we obtain

$$\mathbf{p}_k\,\mathbf{H}\pi^2 \equiv \mathbf{p}_{k^2}\,\pi^2 \equiv \mathbf{v}_{k^2}\,\pi^2 + \mathbf{p}_{k^2}\,\mathbf{H}\pi^3 \,, \tag{10.3}$$

which in turn shows the second order residual as a sum of the rate of profit (at the second power) on third order vertically integrated labour coefficients plus a third order residual containing prices, and itself liable to be split up into two further components. This logical chain may be pursued, step by step, to whatever degree we may choose. By using the same recurring formula (9.4), we obtain

$$\begin{aligned}
\mathbf{p}_{k^2}\,\mathbf{H}\,\pi^3 &\equiv \mathbf{p}_{k^3}\,\pi^3 \equiv \mathbf{v}_{k^3}\,\pi^3 + \mathbf{p}_{k^3}\,\mathbf{H}\,\pi^4 \,, \\
&\;\;\vdots \\
\mathbf{p}_{k^{s-1}}\,\mathbf{H}\,\pi^s &\equiv \mathbf{p}_{k^s}\,\pi^s \equiv \mathbf{v}_{k^s}\,\pi^s + \mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} \,,
\end{aligned} \tag{10.4}$$

where s is a natural number as high as we may choose. Each step may now be substituted back into the previous one, in (10.4), and then in (10.3), (10.2), (10.1), so as to obtain

$$\mathbf{p} = \mathbf{v} + \mathbf{v}_k\,\pi + \mathbf{v}_{k^2}\,\pi^2 + \ldots \mathbf{v}_{k^s}\,\pi^s + \mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} \,. \tag{10.5}$$

There still remains an $(s+1)^{th}$ order residual, but this residual can be made as small as may suit one's purpose by making s as great as is necessary. In the limit, as $s \rightarrow \infty$, the residual vanishes (15)

(15) The $(s+1)^{th}$ order residual, after substitution from recurring formula (9.4), may be written as

$$\mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} = \mathbf{p}\,(\pi\,\mathbf{H})^{s+1} \,.$$

Supposing $\mathbf{p} > \mathbf{o}$, a necessary and sufficient condition for this expression to vanish, as $s \rightarrow \infty$, is $\lim_{s\rightarrow\infty} (\pi\,\mathbf{H})^s = \mathbf{O}$. This is precisely the case if $\pi < \dfrac{1}{|\lambda|_{max}}$. A proof can be given by using the similarity transformation of matrix $\mathbf{H}$

— 16 —

and prices entirely resolve themeselves into an infinite sum of weighted quantities of labour:

$$(10.6) \qquad \mathbf{p} = \mathbf{v} + \mathbf{v}_k \pi + \mathbf{v}_{k^2} \pi + \mathbf{v}_{k^3} \pi^3 + \dots$$

The remarkable upshot of this succession is that at the first round of approximation we find the first order vertically integrated labour coefficients, at the second round of approximation we find the second order vertically integrated labour coefficients, at the third round the third order vertically integrated labour coefficients, and so on. Since these rounds go on to infinity, *all* higher order vertically integrated labour coefficients contribute to the logical process of finding the final solution.

The condition under which the infinite series (10.6) is convergent can be seen immediately upon substitution from (9.2). We obtain

$$(10.7) \qquad \mathbf{p} = \mathbf{v}\,[\mathbf{I} + \pi\,\mathbf{H} + \pi^2\,\mathbf{H}^2 + \pi^3\,\mathbf{H}^3 + \dots],$$

where within square brackets appear in succession all the higher order units of vertically integrated productive capacity, appropriately weighted with the powers of the rate of profit. It is not difficult to see that the series is convergent provided that $\pi < \frac{1}{|\lambda|_{max}} = \Pi$ [16]. Only when $\pi = \Pi$, i.e., when all the purchasing power of prices is absorbed by profits, is the series not convergent, and prices can « command » an infinite quantity of labour. To the opposite extreme is the case in which $\pi = 0$, which makes all profit-weighted addenda vanish; and prices (in terms of « labour commanded ») become equal to the only unweighted addendum in the series — classical « labour embodied ». In between these two extremes, i.e. for $0 < \pi < \Pi$, the series is infinite and convergent. As may be noticed, the series actually corresponds to the well known iterative numerical method for obtaining the inverse of matrix $(\mathbf{I} - \pi\,\mathbf{H})$ which appears in (5.11)

---

called its *Jordan canonical form*, i.e., $\mathbf{F} = \mathbf{V}\,\mathbf{H}\,\mathbf{V}^{-1}$, where $\mathbf{V}$ is a square non-singular matrix and $\mathbf{F}$ is a matrix with all eigenvalues of $\mathbf{H}$ on its main diagonal and either zeros or ones on the diagonal next to the main one. Clearly $\mathbf{F}^s = \mathbf{V}\,\mathbf{H}^s\,\mathbf{V}^{-1}$. It can now be seen that if $\pi < \frac{1}{|\lambda|_{max}}$, all elements of $(\pi\,\mathbf{F})^s$ tend to zero as s tends to infinity. This ensures the tendency of $(\pi\,\mathbf{H})^s$ to $\mathbf{O}$ as s tends to infinity.

[16] This result is an immediate consequence of what is shown in the previous footnote, the series being a geometric one. The convergence of the infinite series (10.7), when $\pi < \frac{1}{|\lambda|_{max}}$, is a particular case of a more general theorem concerning functions of matrices. For a rigorous proof of this more general theorem, see for example: C.C. MacDuffee, « The Theory of Matrices », New York 1946, pp. 97 and ff.; Salvatore Cherubino, « Calcolo delle Matrici », Roma 1957, ch. IV.

In other words, the sum of the infinite series converges to inverse matrix

$$(10.8) \qquad (\mathbf{I} - \pi\,\mathbf{H})^{-1}\,,$$

which means that step-by-step solution (10.6) converges to « exact » solution (5.11).

The notions of higher order vertically integrated labour coefficients have therefore the remarkable property of conferring an economic meaning of high theoretical relevance on each round of approximation to be carried out in the search for the price solution. They resolve the price of every commodity into a sum of profit-weighted quantities of labour. [17]

11. A « DUAL » EXERCISE

Matrix **H** and all its powers have a dual conterpart which, though not essential to the arguments of the present paper, will here be evinced explicitly for the sake of completeness. The analytical framework of the previous pages enables us to proceed very quickly at this stage, as we can start directly with an application that brings out all the dual notions at once.

Suppose that in the economic system considered so far the labour force is growing in time at the steady percentage rate $g > 0$ per annum, i.e.,

$$(11.1) \qquad L(t) = L(0)\,[1 + g]^t\,.$$

And suppose that average *pro-capite* consumption is also constant through time, so that we may write **c** for the column vector of average *pro-capite* consumption coefficients. We have

$$(11.2) \qquad \mathbf{C}(t) = \mathbf{c}\,\mu\,L(t)\,,$$

where μ is the (constant) proportion of active to total population.

We shall consider the problem of finding a solution for the equilibrium (full employement) composition of total production $\mathbf{X}(t)$ in each year t. Of course, $\mathbf{X}(t)$ must include: commodities for consumption, commodities for new investments (i.e., for the expansion at rate g of all fixed and circulating capital goods, whether used for the production of consumption or of investment goods) and commodities for the replacement of all used-up capital goods (whether used up by production of consumption or of investment goods). i.e.,

[17] As may be realized, expression (10.6), by being the iterative solution of (5.11), also represents an iterative solution of Marx's «transformation problem». So Marx was not off the track, after all, when he sensed he could start from «values» and calculate profits directly on them. But he tried to settle the problem in one step, while what is needed is a long iterative process.

— 18 —

$$X(t) = C(t) + g\,A\,X(t) + A^{\ominus}\,X(t)\,. \tag{11.3}$$

This system of equations may of course be solved immediately for $X(t)$. If we follow a slightly round-about way, we obtain

$$X(t) = (I - A^{\ominus})^{-1}\,C(t) + g\,(I - A^{\ominus})^{-1}\,A\,X(t)\,,$$

$$X(t) = [I - g\,(I - A^{\ominus})^{-1}\,A]^{-1}\,(I - A^{\ominus})^{-1}\,C(t)\,. \tag{11.4}$$

We may now define a new matrix G, i.e.,

$$G \equiv (I - A^{\ominus})^{-1}\,A, \tag{11.5}$$

which immediately appears as dual to H. After substitution into (11.4) we may write

$$X(t) = (I - g\,G)^{-1}\,(I - A^{\ominus})^{-1}\,C(t)\,. \tag{11.6}$$

This expression concerning physical quantities is clearly dual to expression (5.11) concerning prices. In general, of course,

$$G \equiv (I - A^{\ominus})^{-1}\,A \neq A\,(I - A^{\ominus})^{-1} \equiv H\,.$$

But G and H have exactly the same eigenvalues. In particular, $\lambda_{max} = |\lambda|_{max}$ is the maximum eigenvalue of both of them.

We may now proceed, as in the previous section, to finding the solution of (11.3) through the alternative procedure of successive approximations. Total production $X(t)$ must certainly contain a batch of commodities, which we may call $X^{I}(t)$, that provide for consumption goods $C(t)$ and for all commodities that go to replace the used-up means of production for producing $C(t)$, i.e.,

$$X^{I}(t) = C(t) + A^{\ominus}\,X^{I}(t)\,,$$

$$X^{I}(t) = (I - A^{\ominus})^{-1}\,C(t)\,. \tag{11.7}$$

If the economic system were stationary (i.e., if $g = 0$), that would be all that is needed; $X^{I}(t)$ would simply coincide with $X(t)$ and this would be the end of the story. But we are supposing $g > 0$. Therefore, another batch of commodities, which we may call $X^{II}(t)$, is needed for *expansion* of the capital goods needed for the production of $X^{I}(t)$ and also for replacement of the capital goods to be used up for $X^{II}(t)$, i.e.,

$$X^{II}(t) = g\,A\,X^{I}(t) + A^{\ominus}\,X^{II}(t)\,,$$

$$X^{II}(t) = g\,(I - A^{\ominus})^{-1}\,A\,X^{I}(t) \equiv g\,G\,X^{I}(t)\,. \tag{11.8}$$

A chain argument has now been started. What has been said for $X^{I}(t)$ must be repeated for $X^{II}(t)$. A third batch of commodities $X^{III}(t)$ is needed for expansion at growth rate g of $X^{II}(t)$ and replacement of the corresponding capital goods, i.e.,

— 19 —

(11.9) $$\mathbf{X}^{III}(t) = g\,\mathbf{G}\,\mathbf{X}^{II}(t) = g^2\,\mathbf{G}^2\,\mathbf{X}^{I}(t)\,.$$

And so the chain argument goes on. A fourth batch of commodities $\mathbf{X}^{IV}(t)$ is needed for the successive round, and then a fifth batch of commodities, a sixth batch, a seventh, and so on to infinity:

(11.10) $$\begin{aligned} \mathbf{X}^{IV}(t) &= g\,\mathbf{G}\,\mathbf{X}^{III}(t) \equiv g^3\,\mathbf{G}^3\,\mathbf{X}^{I}(t)\,, \\ &\vdots \\ \mathbf{X}^{s}(t) &= g\,\mathbf{G}\,\mathbf{X}^{s-1}(t) \equiv g^{s-1}\,\mathbf{G}^{s-1}\,\mathbf{X}^{I}(t)\,. \\ &\vdots \end{aligned}$$

Total production $\mathbf{X}(t)$ clearly consists of the conceptual sum of the infinite serie

(11.11) $$\mathbf{X}(t) = \mathbf{X}^{I}(t) + \mathbf{X}^{II}(t) + \mathbf{X}^{III}(t) + \ldots$$

or, after substitution from (11.7) - (11.10),

(11.12) $$\mathbf{X}(t) = [\mathbf{I} + g\,\mathbf{G} + g^2\,\mathbf{G}^2 + \ldots]\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{C}(t)\,,$$

where in square brackets appears the series of all the powers of matrix $\mathbf{G}$, appropriately weighted with the powers of g. This series in clearly dual to the series in (10.7), while expression $(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{C}(t)$ is dual to expression $\mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{v}$. Again it is not difficult to see that the present series converges to inverse matrix

(11.13) $$(\mathbf{I} - g\,\mathbf{G})^{-1}\,,$$

provided only that $g < \dfrac{1}{|\lambda|_{max}}$, [18] which is exactly the same condition required for convergence of the series in (10.7). As was to be expected, step-by-step solution (11.12) converges to exact solution (11.6).

The problem remains of giving matrix $\mathbf{G}$ an explicit economic interpretation. We have seen in the previous pages that the columns of matrix $\mathbf{H}$ represent the m units of vertically integrated productive capacity — each column i of $\mathbf{H}$ represents the series of heterogeneous commodities directly and indirectly required as *capital good stocks* in the whole economic system in order to produce one physical unit of final good i ($i = 1, 2, \ldots m$). The economic meaning of $\mathbf{G}$ is the exact dual counterpart. Each column j of matrix $\mathbf{G}$ represents the series of heterogeneous commodities directly and indirectly required

(18) The proof may be given along the same lines as those indicated with reference to $\pi < \dfrac{1}{|\lambda|_{max}}$ and matrix $\mathbf{H}$ in footnotes 15) and 16) above.

as *flows* in the whole economic system in order to produce all the stocks of capital goods necessary for one physical unit of commodity j ($j = 1, 2, \dots m$). While $\mathbf{H}$ is a matrix of stocks for the production of flows, $\mathbf{G}$ is a matrix of flows for the production of stocks.

And, of course, the logical process that leads to matrix $\mathbf{G}$ can be applied all over again, in the same way as the logical process leading to matrix $\mathbf{H}$ has been applied all over again in section 9. The flows represented by $\mathbf{G}$ themselves require stocks of capital goods, and the production of these stocks requires (directly and indirectly) the flows represented by $(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{A}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{A} \equiv \mathbf{G}^2$. A further step back yields $\mathbf{G}^3$ and a still further step back yields $\mathbf{G}^4$, and so on. This logical process may be pursued to any higher order as we may choose; each step requiring pre-multiplication by $\mathbf{G}$. All these higher order notions represented by the powers of matrix $\mathbf{G}$, appropriately weighted with the powers of g, then appear in the infinite series (11.12), where they confer a specific economic meaning on the successive rounds of approximation to be carried out in the search for the equilibrium growth solution.

12. Production with fixed capital goods in general

The whole analysis has been carried out so far on the simplifying assumption that a constant proportion of all fixed capital goods drops out of the production process each year. This assumption may now be relaxed. In general, the way in which fixed capital goods wear out may vary widely from one industry to another and from one type of equipment to another. But in principle there is no difficulty in representing analytically any pattern of capital-good wear and tear. All capital goods may be considered, at the beginning of each year, to be entering the production process as particular inputs. Then, when, at the end of the year, they come out of the production process one year older, they may be considered as different commodities jointly produced with the commodities they contribute to produce.

This procedure requires each industry to be decomposed into as many « activities » as there are « years » in which the capital goods are used, each activity representing the same process of production but with a fixed capital good of a different age. Since each activity except one (the final one in which the capital good concerned drops out of the production process) produces jointly with the good that is produced also a capital good of a different age (which is considered as a different commodity), equality is maintained between the number of activities and the number of commodities.

Analytically, if technical coefficients remain constant through time, the technique for the whole economic system may be represented by:

i) a non-negative row vector of direct labour coefficients $\mathbf{a}_{[n]} \equiv [a_{nj}]$, $n = m + 1$, $j = 1, 2, \dots m$, where j stands now for

— 21 —

the j^{th} activity and m for the number of activities (and of commodities);

ii) a non-negative square matrix of commodity-input coefficients $\mathbf{A} \equiv [a_{ij}]$, $i, j = 1, 2, \ldots m$. This matrix includes all capital goods, both circulating and fixed, since all of them are considered as entering the production process as inputs at the beginning of the year;

iii) a non-negative square matrix of commodity-output coefficients $\mathbf{B} \equiv [b_{ij}]$; $i, j = 1, 2, \ldots m$. This matrix represents all commodities existing at the end of each year — consumption goods, and capital goods of all types, new and old.

To complete the notation a convention must be chosen regarding the normalization of all technical coefficients (i.e., regarding the scale to which each unit-activity is referred). And the choice made here is to refer all the coefficients on each column (activity) j of $\mathbf{a}_{[n]}$, $\mathbf{A}$, $\mathbf{B}$, to the physical unit produced of commodity $i = j$ $(i, j = 1, 2, \ldots m)$. This procedure has the convenient property of making all elements on the main diagonal of $\mathbf{B}$ equal to unity, (after suitable re-arrangement of rows and columns), and therefore of allowing us to make use of all notation defined in section 2.

When production takes place with fixed capital goods in general, the physical economic system is thus represented by systems of equations

$$(\mathbf{B} - \mathbf{A})\ \mathbf{X}(t) = \mathbf{Y}(t), \tag{12.1}$$

$$\mathbf{a}_{[n]}\ \mathbf{X}(t) = L(t), \tag{12.2}$$

$$\mathbf{A}\ \mathbf{X}(t) = \mathbf{S}(t), \tag{12.3}$$

and prices by system of equations

$$\mathbf{p}\,\mathbf{B} = \mathbf{a}_{[n]}\,w + \mathbf{p}\,\mathbf{A} + \mathbf{p}\,\mathbf{A}\,\pi\,. \tag{12.4}$$

In the particular case of constant proportion depreciation, the number of activities reduces to one in each industry; output matrix $\mathbf{B}$ reduces to indentity matrix $\mathbf{I}$; and input matrix $\mathbf{A}$, which appears both in (12.1) and on the second addendum of (12.4), reduces to $\mathbf{A}^{\Theta}$. General system of production with fixed capital goods (12.1) - (12.4) reduces to the previously considered particular system (2.1) - (2.4).

13. Generalizations and restrictions

The complications of production with fixed capital goods in general make it no longer possible to give an unambiguous meaning to the notion of « industry ». (Each industry may be made up of many activities each of which has its own labour coefficient and its own unit of direct productive capacity). But the notion of vertically integrated sector remains unaffected by complications. The

— 22 —

whole economic system remains susceptible to being conceptually decomposed into m sub-systems precisely in the same way as has been done in section 4. The only formal difference is that matrix $(\mathbf{B} - \mathbf{A})^{-1}$ takes the place of matrix $(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}$ in expressions (4.1), (4.2), (4.3). Therefore, for production with fixed capital goods in general, the vertically integrated labour coefficients and the physical units of vertically integrated productive capacity come to be defined respectively by the components and by the columns of

$$\mathbf{v} \equiv \mathbf{a}_{[n]} (\mathbf{B} - \mathbf{A})^{-1}, \tag{13.1}$$

$$\mathbf{H} \equiv \mathbf{A} (\mathbf{B} - \mathbf{A})^{-1}, \tag{13.2}$$

which represent a generalization of (4.6), (4.7). Similarly the vertically integrated sectors of higher order come to be defined by

$$\mathbf{v}_{k^{s-1}} \equiv \mathbf{a}_{[n]} (\mathbf{B} - \mathbf{A})^{-1} \mathbf{H}^{s-1}, \tag{13.3}$$

$$\mathbf{H}^{s} \equiv [\mathbf{A} (\mathbf{B} - \mathbf{A})^{-1}]^{s}, \tag{13.4}$$

(where s is any positive natural number), which represent a generalization of (9.2), (9.3).

What becomes more difficult to do, in the case of production with fixed capital goods in general, is to devise a neat way of discriminating between the cases in which the above expressions have and the cases in which they do not have an economic meaning. In the simplified case of the previous pages the procedure is clear. Non-negativity of $\mathbf{A}^{\ominus}$ is sufficient to ensure non-negativity of $(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}$. But here the fact that both $\mathbf{B}$ and $\mathbf{A}$ are non-negative *ex-hypothesis* does not necessarily imply that $(\mathbf{B} - \mathbf{A})^{-1}$, and as a consequence $\mathbf{v}$, $\mathbf{H}$ and $\mathbf{G}$, should also be non-negative. Actually $\mathbf{v}$, $\mathbf{H}$ and $\mathbf{G}$ might indeed contain some negative elements and still make good economic sense. What we can say is that, since prices cannot be negative, $\mathbf{v}$ cannot be accepted as economically meaningful if it contains negative components when the rate of profit is zero. But there is nothing to prevent prices from all being positive, even if some components of $\mathbf{v}$ are negative, when the rate of profit is positive. And in this case a vector $\mathbf{v}$ with some negative components would make perfectly good economic sense. Similarly, we can say that if a particular column k of matrix $\mathbf{H}$ contains some negative components, the production of commodity k alone as a final good would require some activities to be run in reverse, and this would be impossible (and thus would have no economic sense). But commodity k might not be produced as a final good at all, or there might be no necessity to produce it alone (the sub-systems are only conceptual, not real, constructions). And in this case too a matrix $\mathbf{H}$ with some negative components would make perfectly good economic sense. A similar (but dual) argument can be developed for matrix $\mathbf{G}$.

In any case the prices of the m commodities, expressed in ordinary

— 23 —

physical units, continue to be given by the formulations of section 5 above, again with the only difference that more general matrix $(\mathbf{B} - \mathbf{A})$ is to replace $(\mathbf{I} - \mathbf{A}^{\ominus})$, and more general matrix $\mathbf{A}$ to replace $\mathbf{A}^{\ominus}$. Similarly the prices of the *m* composite commodities, expressed in physical units of vertically integrated productive capacity of any order, continue to be expressed by (9.4), (9.5), with the more general definitions of $\mathbf{v}$ and $\mathbf{H}$ given by (13.1), (13.2). Actually, if both $\mathbf{v}$ and $\mathbf{H}$ happen to be non-negative, all remarks made in section 5 hold good in their entirety. In the case in which $\mathbf{v}$ and/or $\mathbf{H}$ do happen to contain some negative elements, what is no longer certain is that prices should remain all non-negative (i.e., economically meaningful) at *all* levels of the rate of profit; and as a consequence that the relation between w and π should always be inverse and monotonic in terms of *all* commodities. However, the remarks made in section 5 on the relationship between the classical notions of « labour commanded » and « labour embodied » continue to hold.

But the most interesting results of all refer to the elaborations of sections 10 and 11, which do continue to hold. The step-by-step solution for prices $\mathbf{p}$ continues to be represented by infinite series (10.6) or (10.7), with the vertically integrated units of productive capacity of all orders $\mathbf{H}$, $\mathbf{H}^2$, $\mathbf{H}^3$, ... and with the vertically integrated labour coefficients of all orders, $\mathbf{v}$, $\mathbf{v}_k$, $\mathbf{v}_{k^2}$, ..., being defined by more general expressions (13.1) - (13.4). The condition of convergence of the series is again the same, i.e. $\pi < \frac{1}{|\lambda|_{max}}$ [19]. Similarly, the step-by-step solution for total production $\mathbf{X}(t)$ continues to be expressed by the infinite series (11.12), with more general matrix $(\mathbf{B} - \mathbf{A})^{-1}$ in the place of $(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}$. The series is again convergent for $g < \frac{1}{|\lambda|_{max}}$.

What must be added here is that we can no longer be certain that the eigenvalue of $\mathbf{H}$ and $\mathbf{G}$ which is maximum in modulus — i.e. $|\lambda|_{max}$ — is also the eigenvalue which is economically relevant. If there exists a $|\lambda|_{max} > \lambda_e$, where λ_e represents the economically relevant eigenvalue, the series (10.7) and (11.12) converge for all π and g smaller than $\frac{1}{|\lambda|_{max}}$, but do not converge for π and $g \geq \frac{1}{|\lambda|_{max}}$. In other words (and with reference to π, for the sake of brevity, since the same thing can be repeated for g), if we define $\pi^* = \frac{1}{|\lambda|_{max}}$, the series (10.7) converges for all rates of profit within the range $0 < \pi < \pi^*$. In those cases in which λ in $|\lambda|_{max}$ is a real and positive number, π^* coincides with Π, and (10.7) converges, as before, for all economically significant rates of profit up to Π (but not at, or

---

(19) See footnotes 15) and 16) above.

— 24 —

beyond, II). In other more complicated cases in which $\pi^* < \pi_e$, where $\pi_e = \frac{1}{\lambda_e}$, π^* becomes the new critical level of the rate of profit. The series (10.7) converges for all rates of profit up to π^*, but not at or beyond, π^*. It is however important to realize that $|\lambda|_{max}$ is finite. Therefore π^* is in any case positive. This means that, from zero upwards (even in the most complicated cases of joint production!), there always exists a range of positive rates of profit within which the series (10.7) is convergent.

The reduction of prices to a sum of weighted quantities of labour is thereby revealed to be a result of great generality. The series

$$(13.1) \qquad \mathbf{p} = \mathbf{v} + \mathbf{v}_k \pi + \mathbf{v}_{k^2} \pi^2 + \mathbf{v}_{k^3} \pi^3 + \ldots$$

where $\mathbf{v}_{k^s} \equiv \mathbf{v}\,[\mathbf{A}\,(\mathbf{B} - \mathbf{A})^{-1}]^s$, clearly represents a generalization of Piero Sraffa's reduction of prices to *dated* quantities of labour (which is only possible in the case of single-product industries). The logical process of infinite successive vertical integration is thus revealed to be more general, and to go much deeper, than the logical process of infinite chronological decomposition. A generalization of this type, with all its theoretical implications, is no doubt one of the most remarkable results of the present analysis.

14. Technical progress

So far in our analysis all technical coefficients have been supposed to be absolutely constant through time. But the notion of a vertically integrated sector is not only unaffected by technical change; it actually acquires greater relevance when technical change is present. In particular the notion of a physical unit of productive capacity, by being defined with reference to the commodity that is produced, continues to make sense, as a physical unit, whatever complications technical change may cause to its composition in terms of ordinary commodities.

If there is technical progress in the economic system, we may suppose, for consistency with our previous analysis, that changes take place at discontinuous points in time. Technical coefficients may be supposed to change at the beginning of each year; and then remain constant during the year. With this convention, the whole previous analysis may simply be re-interpreted as referring to a particular year t. This means that all magnitudes considered in the previous pages must be *dated*. Not only physical quantities $\mathbf{X}(t)$, $\mathbf{Y}(t)$, $\mathbf{S}(t)$, etc., but also prices $\mathbf{p}(t)$, technique $\mathbf{a}_{[n]}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$, and, as a consequence, vertically integrated sectors $\mathbf{v}(t)$, $\mathbf{H}(t)$; $\mathbf{v}_k(t)$, $\mathbf{H}^2(t)$; $\mathbf{v}_{k^2}(t)$, $\mathbf{H}^3(t)$; etc., must be written with a time suffix.

— 25 —

A distinction, however, has to be made at this point between two types of technical progress.

a) *Disembodied technical progress.* We may call 'disembodied' technical progress those improvements that do not affect the technical characteristics of capital goods, and simply enable production of larger physical quantities of commodities out of existing capital goods. Analytically this type of technical progress is expressed by the diminution of some technical coefficients and presents no difficulty. Capital goods, measured in ordinary physical units, remain the same, but their relations to capital goods measured in terms of vertically integrated productive capacities change as time goes on. This means that a particular matrix $\mathbf{H}(t)$ expressing the relation between the two types of units comes into existence for each particular year t; so that an appropriate $\mathbf{H}(t)$ has to be used in each year in order to go from the vertically integrated units to the ordinary ones.

b) *Embodied technical progress.* We may call 'embodied' technical progress those improvements that need to be embodied into specific (new) capital goods. In particular these improvements are supposed to be embodied into such new capital goods that render the old ones either entirely or partially obsolete, in the sense that the old capital goods, even if they continue to be used for the time being, will never be replaced by physical capital goods of the same type when they will be replaced. It must therefore be specified that notation $\mathbf{a}_{[n]}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$ is to be understood as denoting the latest technique for the whole economic system, as this is known at the beginning of year t, so that $\mathbf{v}(t) \equiv \mathbf{a}_{[n]}(t)\,[\mathbf{B}(t) - \mathbf{A}(t)]^{-1}$, $\mathbf{H}(t) \equiv \mathbf{A}(t)\,[\mathbf{B}(t) - \mathbf{A}(t)]^{-1}$ represent the corresponding vertically integrated sectors *as they would be* if the technique of time t had been known in the past and the composition of the capital goods had thereby become evenly balanced. This means that technique $\mathbf{a}_{[n]}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$, and corresponding vertically integrated sectors $\mathbf{v}(t)$, $\mathbf{H}(t)$ represent *hypothetical* magnitudes in this case. The *actual* economic system, if it is to be represented in the same way, requires a different notation; for example we may write $\overline{\mathbf{a}}_{[n]}(t)$, $\overline{\mathbf{A}}(t)$, $\overline{\mathbf{B}}(t)$, to denote the technique which is actually in operation in year t (a mixture of activities of different « vintages »). Then $\overline{\mathbf{v}}(t) \equiv \overline{\mathbf{a}}_{[n]}(t)\,[\overline{\mathbf{B}}(t) - \overline{\mathbf{A}}(t)]^{-1}$, $\overline{\mathbf{H}}(t) \equiv \overline{\mathbf{A}}(t)\,[\overline{\mathbf{B}}(t) - \overline{\mathbf{A}}(t)]^{-1}$ will represent the corresponding *actual* vertically integrated sectors. It goes without saying that all cases considered so far, including that of embodied technical progress, may be regarded as particular cases in which hypothetical and actual vertically integrated sectors happen to coincide.

Of course, both the hypothetical and the actual vertically integrated sectors are relevant — when they are distinct fom each other — but for different purposes. The *hypothetical* vertically integrated sectors are crucial to the determination of prices, as they express the latest technique. The *actual* vertically integrated sectors become relevant for the purpose of representing the physical economic system.

— 26 —

15. The particular case of capital goods produced by labour alone

It becomes rather simple at this point to go back to the multi-sector model of economic growth of *New Theoretical Approach*, and view it as a particular case of the analysis of the previous pages. The assumption that all capital goods are made by labour alone, and that they wear out according to a constant proportion $\left[\frac{1}{T}\right]$, makes it a particular case of the analysis of the first sections of the present paper. And the assumption that technical progress takes place by diminution of all labour coefficients makes it a particular case of disembodied technical progress.

Such a simplified economic system possesses many convenient properties. The vertically integrated labour coefficients for consumption goods are expressed by the sum of the direct labour coefficients plus the quantities of labour required by replacements [20], and the vertically integrated labour coefficients for investment goods are simply expressed by direct labour coefficients. The vertically integrated units of productive capacity for consumption goods are expressed by unit vectors, and those for investment goods by zero vectors. Second-order vertical integration is even simpler. In consumption good industries, the second order vertically integrated labour coefficients for the capital goods (measured in units of vertically integrated productive capacity) coincide with their direct labour coefficients; and in investment good industries the second order vertically integrated labour coefficients are all zero. Finally, the second order vertically integrated units of productive capacity are all represented by zero vectors. The vertically integrated sectors of any higher order are all zero.

In matrix notation, if we denote by $\mathbf{O}_i$ the null square matrix of the i^{th} order, and by $\mathbf{I}_i$ the identity matrix of the i^{th} order, the matrices defined in the previous pages (using the symbols adopted in *New Theoretical Approach*) reduce to the following:

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \\ \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix} \quad ; \quad \mathbf{A}^{\ominus} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \\ \frac{1}{T}\mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix} \quad ;$$

(20) It may be useful, in this respect, to point out a misleading formulation that appears on p. 669 of *New Theoretical Approach*. The vector on the left hand side of equality (VI.3) is written with symbols a_{ni}, $i = 1, 2, \dots n\text{-}1$, which — in the previous chapters — are used to indicate *direct* labour coefficients. What should have been done was to use a new symbol, for example — as in the present analysis — $v_1, v_2, \dots v_{n-1}$.

$$(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \frac{1}{T}\mathbf{I}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix} \;;\; \mathbf{H} \equiv \mathbf{A}\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix};$$

$$\mathbf{H}^2 = \mathbf{O} \quad ; \quad (\mathbf{I} - \pi\,\mathbf{H})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \pi\,\mathbf{I}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$\mathbf{a}_{[n]}(t) = [a_{n1}(t)\; a_{n2}(t) \ldots a_{n,n-1}(t)\;\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)]\,;$$

$$\mathbf{v}(t) \equiv \mathbf{a}_{[n]}(t)\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = [(a_{n1}(t) + \frac{1}{T} a_{nk_1}(t)) \ldots (a_{n,n-1}(t) + \\ + \frac{1}{T} a_{nk_{n-1}}(t))\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)\,]\,;$$

$$\mathbf{v}_k(t) \equiv \mathbf{v}(t)\,\mathbf{H} = [a_{nk_1}(t)\;\; a_{nk_2}(t) \ldots a_{nk_{n-1}}(t)\;\; 0\;\; 0 \ldots 0]\,;$$

$$\mathbf{v}_{k_2}(t) \equiv \mathbf{v}_k(t)\,\mathbf{H} = [0\;\; 0 \ldots 0]\,.$$

By substituting these particular expressions into (5.9) or (10.6), we obtain

$$p_i(t) = \left[a_{ni}(t) + \left(\frac{1}{T} + \pi\right) a_{nk_i}(t)\right] w(t)\,,$$

$$p_{k_i}(t) = a_{nk_i}(t)\; w(t)\,,$$

which are precisely the « solutions » for prices given in the original formulation [21].

---

[21] See *New Theoretical Approach*, p. 597. A more complex case is considered on pp. 598-601, in which the capital goods produced by the investment good industries are supposed to be used both in the consumption good industries and in the investment good industries, given a proportion γ_i, $i = 1, 2, \ldots n\text{-}1$, between their productive capacities for the two types of industries. In this case, the matrices defined here (again by using the symbols of *New Theoretical Approach*, and supposing for notational simplicity that T_k, T, γ, π are all uniform) reduce to the following:

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \gamma\mathbf{I}_{n-1} \end{bmatrix} \quad ; \quad \mathbf{A}^{\Theta} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \gamma\frac{1}{T_k}\mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \frac{1}{T}\,\frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} & \frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$\mathbf{a}_{[n]}(t) = [a_{n1}(t)\;\; a_{n2}(t) \ldots a_{n,n-1}(t)\;\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)]\,;$$

$$\mathbf{A}\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} \equiv \mathbf{H} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \left(1 + \frac{1}{T}\gamma\frac{T_k}{T_k - \gamma}\right)\mathbf{I}_{n-1} & \gamma\frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} \end{bmatrix};$$

— 28 —

16. New analytical possibilities for dynamic analysis

But the use made of vertically integrated sectors in *New Theoretical Approach* has also been aimed at the wider purpose of opening up new possibilities for dynamic analysis.

In the general case of production of all commodities by means of fixed capital goods and of technical progress of the most general type, the relation between ordinary physical capital goods and capital goods in units of productive capacity breaks down at the end of each period and the problem arises of what meaning one can give to the physical operation of replacement of the capital goods. Clearly « replacement » ceases to have any meaningful sense in terms of ordinary physical units. On the other hand, « replacement » does continue to make sense in terms of physical units of productive capacity. Even in the midst of a maze of physical and qualitative changes, we may indeed continue to say that replacement of used-up capital goods has taken place if, at the end of each period, the economic system has recovered the same productive capacities as it had at the beginning.

The analytical consequences of these remarks are far reaching. With technical progress, any relation in which capital goods are expressed in ordinary physical units becomes useless for dynamic analysis. But relations expressed in physical units of productive capacity continue to hold through time, and actually acquire an autonomy of their own, quite independently of their changing composition. At the same time the elaborations of the previous pages provide the way for a return to the ordinary physical units any time that this is necessary, within each period t. Of course a different result will be obtained for each single period.

This property seems to me to confer on the logical process of vertical integration an analytical relevance for dynamic investigations which perhaps has not been completely realized as yet. The vertically inte-

---

$$(\mathbf{I} - \pi\,\mathbf{H}) = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ -\pi\left(1 + \dfrac{1}{T}\gamma\dfrac{T_k}{T_k - \gamma}\right)\mathbf{I}_{n-1} & \left(1 - \pi\gamma\dfrac{T_k}{T_k - \gamma}\right)\mathbf{I}_{n-1} \end{bmatrix};$$

$$(\mathbf{I} - \pi\,\mathbf{H})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \dfrac{\pi}{T}\,\dfrac{\gamma T_k + (T_k - \gamma)T}{T_k - \gamma - \pi\gamma T_k}\,\mathbf{I}_{n-1} & \dfrac{T_k - \gamma}{T_k - \gamma - \pi\gamma T_k}\,\mathbf{I}_{n-1} \end{bmatrix};$$

$$\mathbf{v} = [(a_{n1}(t) + \frac{1}{T}\,\frac{T_k}{T_k - \gamma}\,a_{nk_1}(t)) \ldots (a_{n,n-1}(t) + \frac{1}{T}\,\frac{T_k}{T_k - \gamma}\,a_{nk_{n-1}}(t))$$
$$\frac{T_k}{T_k - \gamma}\,a_{nk_1}(t) \ldots \frac{T_k}{T_k - \gamma}\,a_{nk_{n-1}}(t)]$$

Here again, as can easily be checked, post-multiplication of $\mathbf{v}$ by $(\mathbf{I} - \pi\,\mathbf{H})^{-1}$, and by w, yields the expressions for prices given on p. 600 of *New Theoretical Approach*.

— 29 —

grated sectors seem to belong to that category of synthetic notions which, once obtained, contribute to reduce in many directions the very order of magnitude of the analytical difficulties. An example of this is given after all by the multi-sector model of economic growth, from which the present analysis has started, which has permitted the investigation of a whole series of structural dynamic relations — something which would have been impossible to do with any traditional growth model.

It may not be too unjustified to hope that a better understanding, and a more explicit utilization, of the logical process of vertical integration might help to overcome the widely recognized failure of modern economic theory to come to grips with the analytical difficulties of technical change.

Contributions to Political Economy (1985) **4**, 79–90

COST FUNCTIONS AND PRODUCED MEANS OF PRODUCTION: DUALITY AND CAPITAL THEORY*

NERI SALVADORI

University of Catania

and

IAN STEEDMAN

University of Manchester

A recent development in neoclassical theory — the use of the so-called 'dual' cost, profit and revenue functions — provides a most convenient set of analytical tools by means of which different treatments of the role of produced means of production can be compared and contrasted. The purpose of this paper is precisely to use dual concepts, in particular the cost function, both to present some familiar and some less familiar criticisms of certain kinds of neoclassical results and to show how such concepts are well-suited to bring out some central ideas found in the work of Sraffa.

After introducing the cost function, we shall first provide a 'dual' presentation of the well-known capital theory results of the 1960s. Our principal reason for returning to such well trampled ground is to dispel any possible illusions as to whether those familiar results somehow disappear within the context of a 'dual approach', but an incidental result will be the explicit demonstration that heterogeneous primary inputs are very easily encompassed, something which less acute critics of the 'Cambridge criticism' have sometimes seemed to doubt. To revisit the 1960s is, however, by no means our only purpose; we shall also seek to show why the role of produced inputs and of a positive, uniform rate of interest is not always properly allowed for in standard results in duality theory.

In Part I we shall always be dealing with 'proportional price' economies in which the price vector for commodities, considered as outputs, is strictly proportional to that for the same commodities considered as produced inputs. In Part II, however, we shall try to set out rather formally certain relationships between the dual cost function analyses of proportional-price and non-proportional-price

*We are most grateful to Christian Bidard, John Eatwell, Heinz Kurz and Ulrich Krause for their helpful comments. The first author thanks the Ministero della Pubblica Istruzione, Rome, Italy, and the C.N.R. (the Italian National Research Council; CT. 82.0067.10) for financial support.

0277–5921/85/010079 + 11 $03.00/0

economies, hoping thereby to highlight the contrasts and the connections between production-price analysis and versions of neoclassical analysis which use non-proportional-prices.

We hope to show both that some results of neoclassical duality theory can be questioned and that 'non-neoclassical' eoconomists should not be reluctant to employ the 'dual approach'.

PART I. CAPITAL THEORY

The derivation of a cost function from a production function has been familiar since the early neoclassical theory of distribution; the entrepreneur is assumed to minimise, for each set of input prices, the cost of producing a given bundle of commodities, subject to the technical constraint on the transformation of inputs into outputs. More recently, however, it has been shown that from any well-defined cost function, with the usual properties, it is possible to derive a unique *notional* production function; a production function, that is, which would yield the given cost function as the result of the usual cost minimisation procedure. Moreover, this notional production function will exactly coincide with the actual, original production function whenever the latter is convex; while if this convexity condition is not met, only those sections of the notional and actual production functions which do coincide are relevant to the derivation of the cost functions. Hence all the relevant information about the technical conditions can be summarised equally well by either a production or a cost function, each being derivable from the other. To start directly from a *given* cost function is as good (or as bad) a procedure as to start from a production function. (On duality theory see, in decreasing order of simplicity, Baumol (1977), Varian (1984), Fuss and McFadden (1978)).

Textbook treatments of cost functions distinguish, of course, between Marshallian short-run and long-run cost functions; they also show that, with competitive markets, the supply prices of produced commodities coincide with their marginal costs, i.e., with the Lagrangean multipliers implicit in the cost minimisation problem. By contrast, such treatments are rarely concerned with the question whether those supply prices are proportional to the prices of reproducible inputs. When this condition holds, i.e. when $\bar{p} = kp$ — where $\bar{p}(p)$ is the input (output) price vector and k is a scalar — k can, of course, be interpreted as the 'interest factor' (or as the ratio of that factor to the inflation factor) and the economy will be able to replicate the same relative prices in the following period, under certain significant assumptions. In the absence of such price-proportionality, however, relative prices will probably change from one period to another. It can be argued that this price-proportionality condition is precisely the condition which interested both the classical economists and the early neoclassical economists. (It is, indeed, the condition that prices be 'prices of production'). We shall thus refer to a proportional-prices economy and to a non-proportional-prices economy; much of our analysis will be concerned with the former but Part II will deal specifically with the relations between the two kinds of economy. When dealing with a proportional-prices economy, we shall write $\bar{p}=(1+r)p$, where r is the rate of interest and inflation is ignored.

A dual approach to capital theory

In a simple one-commodity world, the marginal product of 'corn' capital equals the rate of interest or, equivalently, the slope of the 'per worker' production function equals the rate of interest. The dual expression of the same relationship is that the slope of the real wage-rate of interest frontier is equal, at every point, to the 'capital–labour ratio' which would be in use at the corresponding wage rate and interest rate. Our objective, then, is to find the conditions under which, in a multi-commodity, uniform interest rate world, with any number of kinds of labour (or, more generally, primary inputs), the various slopes of the real wage rates–interest rate frontier are equal to the corresponding capital–labour ratios or the corresponding labour–labour ratios. For the most part, we shall consider a stationary economy in which net output always contains the various commodities in the same proportions, whatever the distribution of income. Joint production and fixed capital will both be excluded, in the interests of simplicity.

Unit cost functions and the wages–interest rate frontier

Consider a vector of differentiable unit cost functions given by

$$c=c(m,\bar{p}) \tag{1}$$

where c, m, $\bar{p}$, are vectors of money unit costs, money wage rates (primary input prices), and money rentals on produced inputs, respectively. Since $c(\ )$ is linear homogeneous in $(m, \bar{p})$ and since its partial derivatives give the matrices of primary input use per unit of output, E, and produced inputs per unit of output, A, we have

$$c\equiv mC_m+\bar{p}C_{\bar{p}}\equiv mE+\bar{p}A, \tag{2}$$

where C_m, C_p are matrices of partial derivatives.

In long-period positions, we shall have both $c=p$ and $\bar{p}=(1+r)p$, where p is the money price vector and r the uniform rate of interest, if primary inputs are paid for *ex post* and produce inputs *ex ante*. Then (2) can be written as

$$p=mE+(1+r)pA \tag{3}$$

where E and A are both functions of $(m,\bar{p})$.

The equation

$$F(m,r,\bar{p})\equiv(1+r)^{-1}\bar{p}-c(m,\bar{p})=0 \tag{3a}$$

defines the implicit function $\bar{p}=\bar{p}(m,r)$ in any neighbourhood of a point $(m,r,\bar{p})$ satisfying (3a) and such that $\det[(1+r)^{-1}I-C_{\bar{p}}]\neq 0$. But since this last condition is equivalent to $\det[I-(1+r)A]\neq 0$, it does indeed hold at all points of economic interest other than ($m=0$, $r=$ the maximum rate of interest). Thus if $m>0$, we can solve (3) as

$$\bar{p}=\bar{p}(m,r) \tag{4}$$

and

$$p=f(m,r)\equiv(1+r)^{-1}\bar{p}(m,r) \tag{5}$$

From (4), then, (3) can be written as

$$p=mE(m,r)+(1+r)pA(m,r) \tag{4'}$$

or

$$p=mL(m,r)+rpH(m,r) \tag{4''}$$

where $L\equiv E(I-A)^{-1}$ and $H\equiv A(I-A)^{-1}$. Naturally, (5) may be thought of as the solution to either (4′) or (4″).

Let the semi-positive column vector z represent the standard of value. Multiplying both sides of (5) by z and then dividing through by (pz), and remembering that $f(\ )$ is linear homogeneous in m, we obtain the real wage rates–interest rate frontier

$$f(w,r)z=1, \tag{6}$$

where $w\equiv(pz)^{-1}m$.

'Factor' use and the derivatives of the wage rates–interest rate frontier

Consider

$$mE+(1+r)pA=p$$

where (E,A) are variable. Differentiating totally, we obtain

$$dmE+drpA+(1+r)dpA+[mdE+(1+r)pdA]=dp.$$

But cost minimising choice of processes, in the face of given (m,r,p) in a competitive economy, will ensure that the square bracket is zero; hence

$$dmE + drpA + (1+r)dpA = dp$$

or

$$dmL + drpH + rdpH = dp \tag{7}$$

Let z be the standard of value, as above, and define $l \equiv Lz$; from (7)

$$dwl + dr(pHz) + r(dpHz) = 0 \tag{8}$$

must always hold on the real wage rates-interest rate frontier.

Suppose that $dr = 0$ and that dw_i and dw_j are the only non-zero elements of dw in (8). We see that

$$-l_i dw_i = l_j dw_j + r(dpHz)$$

and hence that

$$-(dw_i/dw_j) = (l_j/l_i) \text{ iff } r(dpHz) = 0.$$

Now, the standard duality theory proposition is that the (absolute) slope of the 'factor price frontier', in any direction, necessarily equals the corresponding 'factor use ratio'. It will be shown below how the slope of the frontier can indeed be interpreted as a (purely hypothetical) 'factor use ratio' but here we must focus on the fact that the conventional result, far from being necessarily correct, holds good only if $r\ (dpHz) = 0$. This is important for the conventional duality analysis, since the supposed identity of frontier slopes with input ratios has embedded in it both the neoclassical theory of demand for inputs and the (demand side of a) marginal productivity theory of 'factor prices'. The conditions for the validity of the standard duality result will surprise no-one familiar with 1960s capital theory; they obtain:

(1) When $r = 0$.

(2) When the changes in w_i and w_j just happen to make $(dpHz) = 0$, i.e. to produce *no* 'revaluation' of capital stock. This requires $dwL(I - rH)^{-1}\ Hz = 0$ which could, at most, only occur by a fluke. The exception, of course, is the case of Hz proportional to z. Whilst this could hold for *some* H it certainly cannot hold in general, since H is a variable in this analysis. (A special case, of little interest, is clearly that in which A and H are *not* variable — the only choice of method being with respect to E and L — *and* the Standard Commodity is used to measure real wage rates. Then $-(dw_i/dw_j) = (l_j/l_i)$ *is* ensured).

Special cases aside, then, $-(dw_i/dw_j) = (l_j/l_i)$ iff $r = 0$. When $r > 0$, the 'slope' of the frontier in the (w_i/w_j) direction does *not* equal the relative (vertically integrated) use of primary inputs i and j. It is to be noted that, even when $r = 0$, the (absolute) slope of the wage rates frontier will not be equal (special cases aside) to the ratio of the *direct* uses of primary inputs i and j.

Having considered the relative uses of different primary inputs, we now turn to the various 'value capital–labour' ratios. If $dr \neq 0$ and dw_i is the only non-zero element of dw, (8) yields:

$$-l_i dw_i = dr(pHz) + r(dpHz)$$

Thus $-(dw_i/dr) = (pHz/l_i) \equiv$ the integrated value of capital per unit of 'labour' i if *and only if* $r=0$ (the fluke $(dpHz)=0$ cases aside). This is, of course, a familiar result from the earlier 'capital theory debates' (even if homogeneous 'labour' was usually assumed there). When $r>0$, $-(dw_i/dr)$ differs from the 'value capital/labour$_i$' ratio to an extent depending on the degree of 'revaluation', dp, of the integrated capital stock vector, Hz.

Hence the $-(dw_i/dw_j) = (l_j/l_i)$ and $-(dw_i/dr) = (pHz/l_i)$ results depend completely on *either* $r=0$ *or* utter flukes yielding zero 'capital revaluation' effects. They are thus subject to precisely the same type of criticism as are results derived from 'production functions' which have the value of capital as an argument.

An interpretation

Rather than deriving (8) from (7) one could derive

$$dmL + drpH = dp(I - rH)$$

or

$$dwl^* + drpk^* = 0 \tag{9}$$

where $l^* \equiv L(I-rH)^{-1}z$ and $k^* \equiv H(I-rH)^{-1}z$. It will be clear that l^* can be interpreted as the *hypothetical* vector of labour inputs which would be required to produce z for consumption *and* maintain steady growth at rate r (hyper-integrated labour inputs at growth rate r); k^* is the corresponding *hypothetical* vector of capital stocks. Thus it is always true, on the frontier, that

$$-(dw_i/dw_j) = (l_j^*/l_i^*), \text{ when } dr = 0 = dw_h, \ \forall\ h \neq i,j,$$

and

$$-(dw_i/dr) = (pk^*/l_i^*), \text{ when } dw_h = 0, \ \forall\ h \neq i;$$

the various slopes *can* be identified everywhere with the (purely hypothetical) hyper-integrated ratios of labour use and of value capital/labour ratios. But unless there is actually steady growth at rate r (or fluke technical conditions prevail) those slopes *cannot* be identified with *actual* ratios of hyper-integrated 'factor' use.

Conclusion

We hope to have shown above that the 'dual approach', as conventionally presented, certainly cannot avoid the substance of the 1960s criticisms of certain aspects of neoclassical theory; those criticisms need only be changed in form in order to bear directly on duality theory results. At the same time, cost functions are more immediately related to wage–interest rate frontiers than are production functions, so that the 'dual approach' provides a most convenient starting point for a capital–theoretic discussion. Apart from showing quite explicitly and simply that 'Cambridge theory' is completely unaffected by the number of primary inputs — which was always true but has nevertheless been doubted by some — our principal specific result, in Part I, has been the following: the conventional duality theory equality between (absolute) 'factor price frontier' slopes and the corresponding 'factor use' ratios is valid *only* if the interest rate is zero or there are no capital revaluation effects. This is just as true with respect to two different primary inputs as it is with respect to 'value capital' and a primary input.

If any analysis based on duality theory should appear to deny the familiar capital theory results, the appearance is just that.

PART II. COST FUNCTIONS

In Part I we simply used differentiable unit cost functions to establish certain capital–theoretic results. Here, by contrast, we present a more formal treatment of the aggregate cost function for a constant returns to scale economy having only a finite number of activities. Our analysis is intended to facilitate a comparison between the linear models widely used in capital theory and the cost function analysis of duality theory. More specifically, while cost functions have normally been defined and studied in the absence of any express assumption about the relationship between the price vector for products and the price vector for produced inputs, we shall derive and discuss the cost function for a proportional-prices economy. This function will not be derived directly, however, but rather via a two-stage procedure, which will clarify both the nature of the proportional prices cost function itself and its relation to the more usual cost function. In the first stage, the prices of produced inputs will be represented by a vector written as $[(1+r)p]$, where r is a rate of interest, but since product prices will not be proportional to p (let alone equal to $(1+r)p$) this first stage argument will differ only in appearance from the conventional duality argument. In the second stage, price proportionality will indeed be enforced and the consequences examined. It will be seen that produced input prices enter as arguments of the conventional cost function and, indeed, enter in just the same way as primary input prices; this reflects the fact that no significant distinction is being made, in the conventional cost function, between produced and non-produced inputs. By contrast, it will be seen that the prices of produced inputs do not appear at all in the proportional-prices cost function; precisely because commodity inputs are here treated as reproducible inputs. Their contributions to total cost are, in effect, 'resolved' into primary input prices and the rate of interest.

The usual cost function: An alternative presentation

Consider first the following cost function, in which it is *not* presupposed that product prices are proportional to the corresponding produced input prices:

$$C(m, r, p, q) = \mathrm{Min}_x[(1+r)pAx + mEx]$$

subject to

$$\begin{aligned} Bx &\geqq q \\ x &\geqq 0 \end{aligned} \tag{10}$$

where A is the material input matrix $(n \times m)$, E is the labour input matrix $(s \times m)$, B is the output matrix $(n \times m)$, x is the vector of activity levels, q is the vector of gross outputs $(n \times 1)$, m, r and p are the money wage rate vector $(1 \times s)$, the rate of interest, and the money price vector $(1 \times n)$, respectively, which entrepreneurs find at the beginning of the period.

It is easily proved that this cost function is

(i) non-decreasing in (m, r, p),
(ii) homogeneous of degree 1 in (m, p), in $(m, 1+r)$ and in q,
(iii) concave in (m, p) and in (m, r).
(iv) continuous in (m, r, p),
(v) differentiable where the solution is unique.

Moreover,

(vi) if the cost-function is differentiable at $(\bar{m}, \bar{r}, \bar{p})$, then the conditional input demands can be obtained in the following way:

$$\text{demand for capital} \equiv \bar{p}A\bar{x} = \frac{\partial C}{\partial r}$$

$$\text{demand for input of commodity } i \equiv u_i A\bar{x} = \frac{1}{1+\bar{r}}\frac{\partial C}{\partial p_i}$$

$$\text{demand for labour } j \equiv u_j E\bar{x} = \frac{\partial C}{\partial m_j}$$

where $\overline{x}$ is the solution of (10) at $(\overline{w}, \overline{r}, \overline{p})$ and u_i is the ith unit vector; thus

$$\dot{C}(m, r, p, q) = \begin{bmatrix} Ex \\ pAx \\ (1+r)Ax \end{bmatrix} \tag{11}$$

where $\dot{C}(.)$ is the vector of the derivatives of $C(.)$ with respect to (m, r, p).

Equation (11) shows that: (a) the vector of the partial derivatives of the cost-function with respect to the prices of the commodities is proportional to the vector of demands for the inputs of commodities; but these vectors are equal if and only if $r=0$: (b) the partial derivative of the cost function with respect to the rate of interest is equal to the value of capital — this is the analogue to the fact that the derivative of an aggregate production function with respect to the value of capital is equal to the rate of interest *if prices are unchanged.*

Equation (11) can easily be derived by using the usual procedure. But in the linear case it can be obtained even more easily when single production prevails, i.e., when each column of matrix B has one and only one positive element, which will be assumed, with no loss of generality, to equal 1. Then,

$$C(m, r, p, q) \equiv [(1+r)p\hat{A}(m, r, p) + m\hat{E}(m, r, p)]q$$

where $\hat{A}(m, r, p)$ and $\hat{E}(m, r, p)$ are respectively an $n \times n$ matrix and an $s \times n$ matrix, both functions of (m, r, p), built up from matrices A and E by taking, for each commodity, a process which produces that commodity at minimal cost. Of course $\hat{A}(.)$ and $\hat{E}(.)$ are homogeneous of degree 0 in (m, p) and in $(m, 1+r)$. It is obvious that $C(.)$ is differentiable where and only where $\hat{A}(.)$ and $\hat{E}(.)$ are constant.

The proportional-prices cost function

The minimal linear programme (10) has a dual maximal programme:

$$\max_u uq$$

subject to (12)

$$uB \leqslant (1+r)pA + mE$$
$$u \geqslant 0$$

The vector u is the shadow price vector or marginal cost vector

$$\left(u = \left[\frac{\partial C}{\partial q_1}, \dots, \frac{\partial C}{\partial q_n} \right] \right)$$

and represents the price vector which will be adopted at the end of the period. If, for a given (m, r), there exists a vector $\hat{p}$ such that the solution of programme (12) with $p=\hat{p}, \hat{u}$, is such that $\hat{u}=\hat{p}$, then the economy will be price-proportional.

It is well known (from the theory of linear programming) that $\hat{p}$ has to satisfy the following conditions:

$$\exists x: \quad \begin{aligned} &Bx \geqslant q \\ &\hat{p}B \leqslant (1+r)\hat{p}A + mE \\ &\hat{p}Bx = \hat{p}q = (1+r)\hat{p}Ax + mEx \\ &x \geqslant 0, \hat{p} \geqslant 0 \end{aligned} \tag{13}$$

System (13) in x and in $\hat{p}$ has a solution (cf. Salvadori, 1980) if

$$\exists\ s \text{ such that } s \geqslant 0, \qquad [B-(1+r)A]s \geqslant q.$$

Moreover, if single production prevails and if, for each r and m, there exists a vector x such that $(B-A)x \geqslant 0$ — which must necessarily be the case in any long period position — then it is well known that there exists a function

$$\hat{p} = p(m, r),$$

which is the unique solution of (13) for each (r, m), with the required long period properties. This function is linear homogeneous and increasing with respect to m, continuous and increasing everywhere and generally differentiable with respect to r (except at a finite number of values); it is locally concave with respect to r everywhere it is differentiable and locally convex elsewhere; it is defined in

$$\{(m, r)/m \geqslant 0, 0 \leqslant r < \mathrm{R}\} \cup \{(0, R)\}$$

where

$$R = \sup\{\rho/\exists s \geqslant 0: [B-(1+\rho)A]s \geqslant q\}.$$

Hence, the proportional-prices cost-function in a linear technology could, in a formal manner, be represented by

$$C(m, r, q) = \mathrm{Min}_x[(1+r)p(m, r)Ax + m\,Ex]$$

subject to

$$\begin{aligned} Bx &\geqslant q \\ x &\geqslant 0 \end{aligned} \tag{14}$$

It is easy to prove that the porportional-prices cost-function is

(i) increasing in m and in r,
(ii) homogeneous of degree 1 in m and in q,
(iii) linear in m and in q, locally concave in r everywhere it is differentiable and locally convex elsewhere,
(iv) continuous everywhere it is defined,
(v) differentiable where $p(m, r)$ is differentiable with respect to r.

Moreover,
(vi) if the cost-function is differentiable at $(\overline{m}, \overline{r})$, then:

$$\frac{\partial C}{\partial r} = p(\overline{w}, \overline{r})A\overline{x} + (1+\overline{r})\frac{\partial p}{\partial r}A\overline{x}$$

$$\frac{\partial C}{\partial m_j} = u_j E\overline{x} + (1+\overline{r})\frac{\partial p}{\partial m_j}A\overline{x}$$

where $\overline{x}$ is the solution of (14) at $(\overline{m}, \overline{r}, \overline{q})$.
Statement (vi) is obtained in the following way:
Set

$$z(m, r) \equiv C(m, r, q) - [(1+r)p(m, r)A\overline{x} + mE\overline{x}];$$

then $z(m, r) \leqslant 0$ and $z(\overline{m}, \overline{r}) = 0$; therefore $z(m, r)$ has an extremum at $(\overline{m}, \overline{r})$.

Thus, in a proportional-prices economy, the demands for capital and for quality j labour can be obtained by differentiation of the proportional-prices cost-function only if

$$(1+r)\frac{\partial p}{\partial r}Ax \text{ and } (1+r)\frac{\partial p}{\partial m_j}Ax$$

are respectively equal to zero. In effect, *only if capital revaluation effects are zero* will the partial derivatives of the cost function equal the corresponding 'input' demands. (It will be noted that the only other case in which such equalities obtain is now $(1+r) = 0$, rather than $r = 0$; this stems from the fact that the output arguments of the cost function used here are *gross* outputs).

Conclusion

The more formal analysis of cost functions presented in this part has drawn out the importance of distinguishing between cost functions which treat produced inputs on the same footing as primary inputs and cost functions which take account of price-proportionality and the reproducibility of commodity inputs. In the former case, it has been shown that the partial derivatives of the cost function, with respect to input prices, yield the input demand functions exactly *only* when the interest rate is zero. In the proportional-prices case, with gross outputs as arguments of the cost function, the conventional duality results hold good *only* if there are no capital revaluation effects (or if $(1+r)=0$). Duality theory results are no more immune to capital-theoretic problems in the finitely-many techniques case than in the infinitely-many case. Moreover, the common presentation of the cost function reveals once again the strong tendency of some theorists to treat produced inputs in just the same way as primary inputs, even though it is by no means inherent in the 'duality approach' that they should be so treated.

PART III. SUMMARY AND CONCLUSION

The 1960s critique of neoclassical theory might be expressed in synthetic form by the assertions that (a) produced inputs enter economic theory in a qualitatively different way from primary inputs and (b) value–capital cannot be treated as 'just another input', except in fluke cases in which changes in wage and interest rates provoke no capital revaluation effects. In both Parts I and II above it has been seen, in various forms, how these two assertions bear with equal force on more recent 'dual' presentations of neoclassical theory. In particular, it is generally *not* true that the (absolute) slopes of wage–interest rates frontiers equal the corresponding input use ratios and *not* true that the appropriate partial derivatives of a cost function necessarily yield the 'input demand functions'.

Cost functions, of course, are not the only 'dual' functions used in production theory. Profit functions and revenue functions are also defined and their various partial derivatives identified with particular economic variables. For reasons of space, it has not been possible here to consider whether the alleged properties of profit and revenue functions are any more robust than those of the wage–interest rates frontier and of the cost function but the reader who has followed the arguments presented above would probably be surprised if they were indeed any stronger. (For an analysis of the profit function, see Salvadori and Steedman, 1984).

Many important results of conventional duality theory are not valid in the presence of produced inputs and a uniform, positive rate of interest. Yet the general approach embodied in duality theory is a most useful one and writers inspired by Sraffa's contributions to economic theory should not hesitate to put it to work for their own purposes.

REFERENCES

Baumol, W. J. (1977). *Economic Theory and Operations Analysis* (4th edn), London, Prentice Hall International.

Fuss, M. and McFadden, D. (eds) (1978). *Production Economics: A Dual Approach to Theory and Applications*, Amsterdam, North Holland.

Salvadori, N. (1980). On a generalized von Neumann model, *Metroeconomica*.

Salvadori, N. and Steedman, I. (1984). *Duality Theory and Produced Means of Production*, University of Manchester, Department of Economics, Discussion Paper No. 41.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*, Cambridge, C.U.P.

Varian, H. R. (1984). *Microeconomic Analysis* (2nd edn), New York and London, W. W. Norton.

Zeitschrift für Nationalökonomie 36 (1976), 21—48

Relative Prices as a Function of the Rate of Profit: A Mathematical Note

By

Bertram Schefold, Frankfurt

(Received December 17, 1975)

1. A New Mathematical Theorem

The reswitching debate has made it obvious that prices are in general complicated functions of the rate of profit in single product systems of the Sraffa[1] type:

$$(1+r)\,Ap + wl = p^2.$$

It is well known that productive, indecomposable ("basic") single product systems possess a maximum rate of profit $R>0$ such that the vector of prices expressed in terms of the wage rate $\hat{p}=p/w$ is positive and rises monotonically for $0 \leqslant r < R$, tending to infinity for $r \to R$. The difficulty in analysing such single product systems does not consist in the triviality that prices in terms of the wage rate are an increasing function of the rate of profit, rather it is due to the fact that these prices rise with different "speeds". Relative prices are constant only in one case: when prices are proportional to labour values, i. e. to prices at $r=0$; in general, relative prices deviate from relative labour values for positive rates of profit. (Of course, all these "movements" of prices in function of the rate of profit are purely hypothetical).

---

[1] P. Sraffa: Production of Commodities by Means of Commodities, Cambridge 1960.

[2] A is an indecomposable non-negative (n, n)-matrix, l the labour vector, p the vector of relative prices, r the rate of profit, w the wage rate. The coefficient a_i^j of the matrix A denotes the amount of commodity j required to produce one unit of commodity i. The system is supposed to be productive, i. e. $eA \leq e$ where e is the summationvector $e=(1, 1, \ldots, 1)$.

In this article, a mathematical transformation of the price equations is proposed which makes the functional dependence of the price vector on the rate of profit more explicit. This transformation has several economic applications which will be discussed below. The analysis of the origin and the extent of Wicksell effects is improved, it will also be argued that reasonable assumptions about technology do not allow us to exclude the possibility of reswitching. In this respect, this article represents a criticism of Kazuo Sato's attempt to prove that reswitching is empirically irrelevant[3].

Finally, it is proposed to replace the capital-labour ratio which is dimensionally hybrid by the capital-wage ratio which is dimension-free and not subject to perverse Wicksell effetcs.

In order to exhibit the whole generality of the theory we replace the usual single product system by a joint production system

$$(1+r)\ Ap + wl = Bp$$

where B is a non singular output matrix; the joint production system is supposed to be productive, i. e. $e\,A \leqslant e\,B$.

The important properties of the input output matrix of a single product system are summarized in the spectrum of its eigenvalues. However, since the maximum rate of profit has a more direct economic interpretation than the corresponding eigen-value of the input output matrix of a single product system, we consider the roots of the equation det $[B-(1+r)\ A]=0$ instead of looking for the roots of the equation det $(\lambda B - A)=0$. In slight modification of the conventional terminology, we call a root of det $[B-(1+r)\ A)=0$ semi-simple if rank

$$rk\ [B-(1+r)\ A] = n-1.$$

Whether R is a simple root or not: if R is semi-simple there is up to a scalar factor one and only one "eigenvector" q with $q[B-(1+R)A]=0$. We now assume that det $(B-A)\neq 0$ — a condition which is always fulfilled in productive single product systems. It implies that every vector c of final consumption is producible with non negative activity levels provided $c\,(B-A)^{-1} \geqslant 0$. For the reasons explained below we also assume that det $A \neq 0$. We then have[4]

[3] Kazuo Sato: The neoclassical postulate and the technology frontier in capital theory; The Quarterly Journal of Economics *88* (1974), pp. 353—384.

[4] The non mathematical reader may skip the theorem together with its proof. An alternative proof of the theorem is to be found in B. Schefold:

Theorem 1.1.

Let $R_1,\ldots,R_t$ be the roots of $\det[B-(1+r)A]=0$ with multiplicities $s_1,\ldots,s_t$. The price vector $\hat{p}(r)$ assumes n linearly independent values[5] $\hat{p}(r_1),\ldots,\hat{p}(r_n)$ at any n different rates of profit $r_1,\ldots,r_n$ $(r_i \neq r_j,\ r_i \neq R_j)$, if all roots $R_1,\ldots,R_t$ of the equation $\det[B-(1+r)A]=0$ are semi-simple and if $q_i\, l \neq 0$, $i=1,\ldots,t$, for the associated eigenvectors q_i. Conversely, if one root $\bar{R}$ is not semi-simple or if $\bar{q}_i\, l=0$ for some $\bar{q}_i$, it follows that $\hat{p}(r_1),\ldots,\hat{p}(r_n)$ are linearly dependent for any $r_1,\ldots,r_n$ $(r_i \neq R_j)$ and there will be a vector $\bar{q}$ such that $\bar{q}\hat{p}(r) \equiv 0$ for all r. If $\bar{q}_i$ or $\bar{R}$ is real, there is a real vector $\bar{q}$ with $\bar{q}\hat{p}(r) \equiv 0$ for all r.

Proof: Let s_i be the multiplicity of R_i, R_i semi-simple. We have $\sum_{i=1}^{t} s_i = n$, because $\det A \neq 0$. The roots R_i of $\det[B-(1+r)A]=0$ are the same as those of $\det[I-rA(B-A)^{-1}]$. No root is equal to zero. According to Jordan's theory of Normal Forms[6] (applied to the matrix $A(B-A)^{-1}$), there exist n linearly independent vectors

$$q_{i,1},\ldots,q_{i,s_i};\quad i=1,\ldots,t;$$

with

$$q_{i,1}=q_i,$$

$$q_i = R_i\, q_i\, A(B-A)^{-1},$$

$$q_{i,\sigma}=R_i\, q_{i,\sigma}\, A(B-A)^{-1} - R_i\, q_{i,\sigma-1};\quad \sigma=2,\ldots,s_i.$$

It follows that

$$q_{i,\sigma}[I-rA(B-A)^{-1}] = q_{i,\sigma}\left(1-\frac{r}{R_i}\right) - r q_{i,\sigma-1};\ \sigma=2,\ldots,s_i;$$

and this formula holds for $\sigma=1,\ldots,s_i$, if we define $q_{i,0}=0$ for all i.

Eine Anwendung der Jordanischen Theorie der Normalformen, submitted to Zamp (Zeitschrift für angewandte Mathematik und Physik).

[5] Here and in all of what follows, the price vector is considered as a function of the rate of profit, hence as a *curve* in n-dimensional space in function of the variable r. $\hat{p}(r)$ is said to assume n linearly independent values at the rates of profit $r_1\ldots,r_n$, if the vectors $\hat{p}(r_1),\ldots,\hat{p}(r_n)$ are linearly independent.

[6] See e. g. W. Gröbner: Matrizenrechnung, Mannheim 1966, pp. 201—205.

With this we get

$$
\begin{aligned}
q_{i,\sigma}(B-A)\,\hat{p}(r) &= q_{i,\sigma}(B-A)\,[B-(1+r)A]^{-1}\,l \\
&= q_{i,\sigma}\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i}{R_i-r}\,q_{i,\sigma-1}\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i}{R_i-r}\;\frac{R_i}{R_i-r}\,q_{i,\sigma-1}\,l \\
&\quad + \frac{(R_i\,r)^2}{(R_i-r)^2}\,q_{i,\sigma-2}\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i^2}{(R_i-r)^2}\,q_{i,\sigma-1}\,l + \dots \\
&\quad + \frac{R_i}{R_i-r}\left[\frac{r\,R_i}{R_i-r}\right]^{s_i-1} q_i\,l;
\end{aligned}
$$

$$\sigma=2,\dots,s_i \quad i=1,\dots,t;$$

$$q_i\,(B-A)\,\hat{p}(r) = \frac{R_i}{R_i-r}\,q_i\,l.$$

Define (x' denotes the transposed of vector x)

$$
\begin{gathered}
Q = [q'_{1,1},\dots,q'_{1,s_1},\dots,q'_{t,1},\dots,q'_{t,s_t}]', \\
T = Q\,(B-A).
\end{gathered}
$$

The vector $v(r) = T\hat{p}(r)$ assumes in n points $r_1,\dots,r_n$ ($r_i \neq r_j$, $r_i \neq R_j$) n linearly independent values, if and only if $q_i\,l \neq 0$; $i=1,\dots,t$. The necessity of this latter condition is obvious, for $q_i\,(B-A)\,\hat{p}(r) \equiv 0$ if $q_i\,l = 0$. To verify that $q_i\,l \neq 0$ is sufficient, consider the matrix

$$U = [u(r_1),\dots,u(r_n)]$$

where $u(r) = \det\,[B-(1+r)\,A]\;v(r)$. The components of the vector $u(r)$ are denoted by $u_{i,\sigma}$ where $i=1,\dots,t$; $\sigma=1,\dots,s_i$. We have

$$
\begin{aligned}
u_{i,\sigma} = [R_i\,(R_i-r)^{s_i-1}\,q_{i,\sigma}\,l + \dots + r^{\sigma-1}\,R_i^{\sigma}\,(R_i-r)^{s_i-\sigma}\,q_i\,l]\,\times \\
\times \prod_{j\neq i}\,(R_j-r)^{s_j}.
\end{aligned}
$$

We show that U is not singular by showing that $x=0$ for any vector x such that $xU=0$. Since each row of U consists of the values at n points of a polynomial of degree $n-1$, and since the values at n points determine a polynomial of degree $n-1$ fully, $xu(r)=0$ for $r=r_i$, $i=1,\dots,n$, implies $xu(r)\equiv 0$ for all r. Denoting the components of x in the same way as those of u we find at once that x_{i,s_i} $(i=1,\ldots,t)$ must be zero because $u_{i,s_i}(R_i)\neq 0$ while $u_{j,\sigma}(R_i)=0$ for $(j,\sigma)\neq(i,s_i)$. If $s_i>1$, we must have

$$\sum_{(j,\sigma)\neq(i,s_i)} x_{j,\sigma}\frac{u_{j,\sigma}}{R_i-r}\equiv 0.$$

Hence $x_{i,s_i-1}=0$ since $\lim\limits_{r\to R_i}\dfrac{u_{i,s_i-1}}{R_i-r}\neq 0$ while $\lim\limits_{r\to R_i}\dfrac{u_{j,\sigma}}{R_i-r}=0$ for $(j,\sigma)\neq(i,s_i)$ and $(j,\sigma)\neq(i,s_i-1)$.

Continuing the induction one obtains that $x=0$. Thus U is non-singular and

$$\hat{p}(r)=T^{-1}v(r)$$

assumes n linearly independent values at n different points, if the R_i are semi-simple, and if $q_i l\neq 0$, $i=1,\ldots,t$. The necessity of R_i being semi-simple remains to be shown. Suppose R is a multiple root of $\det[B-(1+R)A]=0$ and $rk\,[B-(1+R)A]<n-1$. There are then two linearly independent q_1, q_2 with

$$q_i[B-(1+R)A]=0;\quad i=1,2;\quad q_i l\neq 0.$$

We get as above

$$q_i(B-A)(B-(1+r)A]^{-1}l=q_i[I-rA(B-A)^{-1}]^{-1}l=\frac{R}{R-r}q_i l;\ i=1,2.$$

If $q_2 l=0$, define $\bar{q}=q_2(B-A)$. Otherwise, we have

$$(q_1-\lambda q_2)(B-A)\,\hat{p}(r)\equiv 0,\ \lambda=\frac{q_1 l}{q_2 l},$$

identically in r with $\bar{q}=(q_1-\lambda q_2)(B-A)\neq 0$, which is impossible, if $\hat{p}(r)$ assumes n linearly independent values in any n points. $\bar{q}$ is a real vector, if R is real.

q. e. d.

It remains to discuss how the statements of the theorem are affected if A is a singular matrix.

There is only one relevant economic reason why A could be singular: if the system contains pure consumption goods, entire columns of A will be zero. More generally, if the rank of A is $n-z$, there will be z vectors $q_1,\ldots,q_z$ with $q_i A=0$ so that

$$q_i\,(B-A)\,[B-(1+r)\,A]^{-1}\,l=q_i\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l=q_i\,l.$$

It follows, as in the proof above, that there will be a vector q

with $\bar{q}\hat{p}(r) \equiv 0$ if $z \geqslant 2$ and/or $q_i l = 0$ for some i. Hence, systems with rank $A < n-1$ are not regular in the sense of the following section.

2. Values and Prices: The Rule and the Exception

If a Sraffa joint production system with det $A \neq 0$, det $(B-A) \neq 0$ has only semi-simple characteristic roots and if none of its eigenvectors is orthogonal to its labour vector, we shall call it *regular*[7]. The theorem then says that the price vector of regular Sraffa systems with n commodities and n industries varies in such a way with the rate of profit that it assumes n linearly independent values at any n different levels of the rate of profit. This means that the price vector of a regular Sraffa system is not only not constant, but its variations in function of the rate of profit result in a complicated twisted curve such that the n price vectors belonging to n different levels of the rate of profit r_i span a $(n-1)$-dimensional hyperplane which never contains the origin (provided $r_i \neq R_j$).

Irregular Sraffa systems on the other hand (i. e. those which have a characteristic equation with a multiple root and/or an eigenvector which is orthogonal to the labour vector) are such that the n price vectors taken at n different levels of the rate of profit can *never* be linearly independent. We must ask ourselves: are regular systems the exception or the rule? What is the economic interpretation of the theorem? What economic interpretation do the exceptions have?

---

[7] We note the following corollary:

Corollary: The systems considered above are regular if and only if the vectors $l, Al, \ldots, A^{n-1} l$ are linearly independent.

Proof: If $q_i l = 0$ for some eigenvector q_i of A, the matrix $F = (l, Al, \ldots \ldots, A^{n-1} l)$ is not regular, since $q_i F = 0$. If $q_i l \neq 0$, $i = 1, \ldots, t$, and if R is not a semi-simple root of the characteristic equation det $[I-(1+r)A] = 0$, there are at least two eigenvectors q_1, q_2 associated with R such that $\bar{q}F = 0$ where $\bar{q} = q_1 + \mu q_2$, $\mu = -(q_1 l / q_2 l)$. Conversely, if the system is regular, and if m is the maximum number such that the vectors $l, Al, \ldots \ldots, A^m l$ are linearly independent, suppose that the subspace $R^{m+1} \subset R^n$ spanned by the vectors $l, Al, \ldots, A^m l$ had dimension $m+1 < n$.

Since the matrix $I-(1+r)A$ maps R^{m+1} onto itself, with det $[I-(1+r)A] \neq 0$, this would imply that $p(r) = [I-(1+r)A]^{-1} l \in R^{m+1}$ for *all* $r \neq R_i$, $i = 1, \ldots, t$, which contradicts theorem 1.1 above.

(This corollary is used in B. Schefold: "Nachworte", section 6, in P. Sraffa, "Warenproduktion mittels Waren", Berlin 1976, pp. 131—226.)

First of all, it is easy to see that the regular systems are the rule from a mathematical point of view because even multiple roots are mathematically exceptional and because an eigenvector will only by coincidence be orthogonal to the labour vector. To put it in more precise terms: One can easily prove that the set of irregular Sraffa systems with (n, n)-input-output-matrices is of measure zero in the set of all Sraffa systems with the same number of commodities and industries.

But this observation taken by itself does not mean much. The set of all semi-positive decomposable (n, n)-matrices is also of measure zero in the set of all semi-positive (n, n)-matrices, and yet it is quite clear that the analysis of the "exceptional" decomposable matrices is of greatest economic interest, although they are more difficult to handle than indecomposable matrices. There is an excellent economic reason why decomposable systems are important: pure consumption goods and other non basics exist; therefore decomposable systems exist.

I should like to argue that matters are quite different with irregular systems. I believe that there is no economic reason why real systems should not be regular or why irregular systems should exist in reality; irregularity is only a fluke, or, at best, an approximation. But there is one kind of irregular system which is very useful to the economists because it provides a simple abstraction from some of the more complicated properties of regular systems. *Die Ausnahme bestätigt die Regel* — the exception confirms the rule. By considering the exception we learn why regular systems are the normal case. The extreme form of an irregular system is one where relative prices are constant, i. e. equal to relative labour values.

Relative prices will be constant and equal to relative labour values if (supposing we are dealing with single product systems)

$$(1+R)\,Al = l$$

i. e. if the labour vector happens to be a right hand eigenvector of the input output matrix. All left hand eigenvectors not associated with the characteristic root R will then be orthogonal to l, because for any left hand eigenvector $\bar{q}$ associated with an eigenvalue $\bar{R} \neq R$, we get

$$(1+\bar{R})\,\bar{q}\,Al = \bar{q}l = (1+R)\,\bar{q}\,Al$$

therefore

$$\bar{q}\,Al = \bar{q}l = 0.$$

It follows from the proofs of our theorem above that

$$q_{t,\sigma_t}(B-A)\,\hat{p}\,(r)\equiv 0$$

for all q_{t,σ_t} except the eigenvector associated with $1+R$. This means that $\hat{p}\,(R)$ is a scalar multiple of a constant vector which is a somewhat roundabout proof that relative prices are equal to relative labour values in this case. (The direct proof is obvious.)

Conversely: if prices are equal to values, the price equation must hold at all rates of profit with constant prices. Putting formally $r=-1$, we get $w\,(-1)\,l=p$. Inserting this into the price equation, we obtain $(1+r)\,Aw\,(-1)\,l+w\,(r)\,l=w\,(-1)\,l$, which implies that l is an eigenvector of A. Because of $l>0$, this eigenvector must belong to the eigenvalue corresponding to the maximum rate of profit. Thus, the condition $(1+R)\,Al=l$ is necessary and sufficient for values being equal to prices in a productive and indecomposable single product system.

If relative prices are equal to relative labour values and if absolute prices are expressed in terms of a commodity, it does not matter which of the commodities is taken as a numéraire; the wage rate will always be related to the rate of profit by a simple linear relationship, because the price in terms of the commodity numéraire of total income Y and of capital K is then constant so that the sum of wages W and profits P can be written as

$$Y=W+P=W+rK$$

therefore

$$W=Y-rK.$$

We verify that if

$$(1+R)\,Al=l$$

the price equation

$$(1+r)\,Ap+wl=p$$

is fulfilled for

$$p=(1+1/R)\,l$$

and

$$w=1-r/R$$

with p being the sum of direct labour l and indirect labour $(1/R)\,l$. By assuming that prices are equal to labour values, one abstracts from the problem of relative prices and focuses attention on the problem of determining the relation between the distribution of income (wages or a share in income) and the rate of profit: An increase in the rate of profit engenders a proportionate diminution of the wage rate and, since the level of employment is given, of wages.

If prices are not equal to values, we get the same simple relationship between wages and the rate of profit only if prices are expressed in terms of a suitable average of commodities: Mr. Sraffa's "Standard Commodity". There is a unique positive eigenvector q associated with the maximum rate of profit in an indecomposable ("basic") single product system, such that

$$(1+R)\, qA = q.$$

Hence

$$q\,(I-A)\,p = q\,(rAp + wl) = (r/R)\, q\,(I-A)\,p + wql$$

i. e.

$$1 = w + r/R$$

as above if prices are expressed in terms of the "Standard Commodity" $[q\,(I-A)\,p=1]$ and if the eigenvector q is normalized so that $ql=1$. In other words, if prices are expressed in terms of a "standard commodity", we can abstract from the complications arising from relative prices and obtain in the general case a linear relationship between wages and the rate of profit which is of the same form as that holding for all commodity price standards in systems where prices equal values. (Total income, however, is not constant in terms of the standard commodity, unless the economy itself is in standard proportions.)

This fact has often been commented upon. What we now learn is that in between the most extreme and most "exceptional" case where prices are equal to values and the "regular" case where prices vary along a twisted curve in function of the rate of profit, there are intermediate cases where (possibly real) vector $\bar{q}$ exists, such that $\bar{q}\hat{p}\,(r) \equiv 0$ so that at least one of the components of the price vector is a linear function with constant coefficients of the other components of the price vector. In formulas: $\bar{q}\hat{p}\,(r) \equiv 0$ implies

$$\hat{p}_i\,(r) = 1/\bar{q}_i\,(\bar{q}_1\hat{p}_1 + \ldots + \bar{q}_{i-1}\hat{p}_{i-1} + \bar{q}_{i+1}\hat{p}_{i+1} + \ldots + \bar{q}_n\hat{p}_n)$$

where $\bar{q}_i \neq 0$.

But the price of one commodity can only be a linear function of the prices of the other commodities if there is an inner technical relationship between the processes of production of the commodities. (If prices equal values, the technical relationship must take the form $(1+R)\,Al=l$). I cannot think of any economic reason why such a relationship should exist. This in general confirms that irregular systems are exceptional and interesting only in so far as they allow to abstract from the complications of regular systems[8].

Going a little further, we may conlcude that the normal case is one where all roots of the system are simple and not only semi-simple because the set of input-output matrices with multiple roots is of measure zero in the set of all input-output matrices of given order. In the case of simple roots the proof of our theorem yields that the vector of prices in terms of the wage rate $\hat{p}$ may be written as

$$\hat{p}\,(r) = S \begin{bmatrix} \frac{R_1}{R_1 - r} q_1 l \\ \vdots \\ \frac{R_n}{R_n - r} q_n l \end{bmatrix}$$

where $S = T^{-1}$ is a non singular matrix, where the $R_1, \ldots, R_n$ are the n distinct roots of the characteristic equation of the system, and where $q_i l \neq 0$, $i = 1, \ldots, n$. This formula shows the functional dependence of the price vector on the rate of profit in a (from the mathematical point of view) simple and explicit form: The n-dimensional complex space C^n is mapped onto itself by the matrix T in such a way that each component of $\hat{p}\,(r)$ is a simple hyperbola in function of the parameter r with a singularity at $r = R_i$.

3. Uniqueness of the System Yielding a Given Vector of Prices

Regular systems are important because the complicated behaviour of their prices implies that the technique does not only determine prices in function of the rate of profit, but that the converse is also true: If the price vector is given at $n+1$ different levels of the rate of profit, there is essentially only one technique which is compatible with those prices. The result derives from the following three theorems.

[8] Systems where a labour theory of value holds have been discussed innumerable times. Perhaps it is instructive to give an example of a single product system where one of the roots is multiple. For the input matrix

$$A = \begin{Bmatrix} 1/3 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/3 \end{Bmatrix}$$

we obtain in det $[I - (1+r)\,A] = 0$ a simple root (maximum rate of profit) if $r = 1/5$ and a double root for $r = 5/4$. This double root is not semi-simple.

Theorem 3.1:

Let two (n, n) joint production systems be given with input matrices A, F, output matrices B, G and labour vectors l, m respectively. If and only if the vector of relative prices in terms of the wage rate is the same for both systems at every level of the rate of profit, the two systems are related by the equations

$$G = MB + Y$$
$$F = MA + Y$$
$$m = Ml$$

where M is a non singular (n, n)-matrix and where $Y\hat{p}(r) \equiv 0$ for all r.

Proof: Define $M = (G - F)(B - A)^{-1}$ and $Y = F - MA$. From

$$l = [B - (1+r)A]\hat{p}$$
$$Ml = M[B - (1+r)A]\hat{p}$$

and

$$m = [G - (1+r)F]\hat{p}$$

we obtain

$$m = M[B - (1+r)A]\hat{p} - rY\hat{p},$$

therefore

$$m = Ml - rY\hat{p}(r).$$

Hence $m = Ml$ and $Y\hat{p}(r) \equiv 0$. The converse is obvious.

q. e. d.

Theorem 3.2:

If prices of two joint production systems coincide at $n+1$ different levels of the rate of profit, they coincide everywhere.

Proof: If the equation (using the notation of the previous theorem)

$$[G - (1+r)F][B - (1+r)A]^{-1} = m$$

holds in $n+1$ points, we also have in $n+1$ points

$$(G - F - rF)(B - A - rA)_{\mathrm{Ad}}\, l = \det[B - (1+r)A]\, m$$

where the subscript *Ad* means the adjoint of the corresponding matrix. On both sides of the equation we have a vector of polynomials of degree n. Since the polynomials coincide in $n+1$ points, they coincide everywhere.

q. e. d.

Since the output matrices of two single product systems of the same order are trivially identical (therefore $M=I$) and since $Y\hat{p}(r)\equiv 0$ implies $Y=0$ if the system is regular, we get at once

Theorem 3.3:

If prices of two regular single product systems coincide at $n+1$ levels of the rate of profit, the two systems are identical.

There are examples of irregular systems which are different and yet yield the same prices at all rates of profit. One obtains several well known results as corollaries of theorem 3.2, e. g. that relative prices in two sector systems are monotonic functions of the rate of profit, or more importantly that two (n, n) single product systems cannot have more than n switchpoints in common, if they are different (a switchpoint is a level of the rate of profit where all prices of two systems — and not only just the real wage — coincide).

This would suggest that two systems must be the more similar the more switchpoints they have in common, or, to put it the other way round, it would seem quite logical from a mathematical point of view to suppose that two systems which are really different cannot have two switchpoints in common, except by a fluke. However, we shall prove that reswitching is not a *mathematical* exception.

4. Reswitching and the Technology Set

The term reswitching has not always been used in the same sense. We shall solely consider the case where only the method of production for one of the commodities in the system is subject to change, i. e. where e. g. the techniques for the production of commodities $2, \ldots, n$ are given and fixed while alternative methods are available for the technique used in the production of commodity 1. Reswitching then means the possibility that a technique used in the production of commodity 1 may be eligible at two different levels of the rate of profit, separated by ranges of the rate of profit where different techniques are eligible.

If only one alternative technique exists, this may be formalized as follows: Let a productive, indecomposable single product system with input matrix A and labour vector l be given. The method of production for commodity i is (a_i, l_i), $i=1, \ldots, n$, where a_i is the vector of physical inputs and l_i the labour input to process i. Reswitching will take place if there are two rates of profit r_1, r_2 where a second technique for the production of commodity 1 (denoted by input vector a_0 and labour input l_0) is as profitable as the original technique (a_1, l_1). I. e. the equation $(1+r)\, a_0 \hat{p} + l_0 = (1+r)\, a_1 \hat{p} + l_1$ must hold at two rates of profit r_1, r_2.

It is useful to begin the discussion with this narrowest possible definition of reswitching.

The condition for reswitching can be rewritten as

$$(a_1 - a_0)\,(1+r)\,\hat{p}\,(r) + (l_1 - l_0) = 0.$$

Reswitching will therefore take place if a technique (a_0, l_0) exists such that $c = (a_1, l_1) - (a_0, l_0)$ is orthogonal to the $(n+1)$-column vector

$$\tilde{p}\,(r) = \begin{bmatrix} (1+r)\,\hat{p}\,(r) \\ 1 \end{bmatrix}$$

for two different rates of profit with

$$(a_0, l_0) = (a_1, l_1) - c \geqq 0.$$

Whether reswitching takes place will thus depend on the availability of an alternative technique for the production of commodity 1 on the one hand and on the shape of the curve $\tilde{p}\,(r)$ on the other. We discuss these two in turn beginning with a theorem about $\tilde{p}\,(r)$.

Theorem 4.1:

$\tilde{p}\,(r)$ takes on $n+1$ linearly independent values in $(n+1)$-dimensional space at $n+1$ different rates of profit r_i (where $r_i \neq R_j$ for all i, j), if and only if the system is regular.

Proof: Using the notation of theorem 1.1 one defines the $(n+1)$-columnvector

$$\bar{u}\,(r) = \det\,[B - (1+r)\,A] \begin{bmatrix} (1+r)\,T\,\hat{p}\,(r) \\ 1 \end{bmatrix}$$

and the matrix

$$\bar{U} = [\bar{u}\,(r_1), \ldots, \bar{u}\,(r_{n+1})].$$

As in the proof of theorem 1.1 one shows firstly that $\bar{U}$ is singular if and only if there is a vector $x \neq 0$ such that $x\,\bar{u}\,(r) \equiv 0$ and secondly that the first $n-1$ components of the vector x must be zero. It is then clear that the last component of x vanishes as well. The rest is analogous to the proof of theorem 1.1.

q. e. d.

This theorem is closely related to the following: If a system is not regular, the price vector moves always within one fixed $(n-1)$-dimensional hyperplane containing the origin. But if the system is regular we have:

Theorem 4.2:

The price vectors $\hat{p}(r_i)$ belonging to $n+1$ different rates of profit $r_1, \ldots, r_{n+1}$, $r_i \neq R_j$ for all i and j, are never on the same $(n-1)$-dimensional hyperplane in n-dimensional space.

Proof: The $n+1$ points $\hat{p}(r_i)$, $i=1, \ldots, n+1$ are on a $n-1$-dimensional hyperplane in n-dimensional space if and only if there is a vector $(\lambda_1, \ldots, \lambda_{n+1}) \neq 0$ such that $\sum_{i=1}^{n+1} \lambda_i \hat{p}(r_i) = 0$ with $\sum_{i=1}^{n+1} \lambda_i = 0$.

This will be the case if and only if the matrix

$$\bar{U} = \begin{bmatrix} \hat{p}(r_1), \ldots, \hat{p}(r_{n+1}) \\ 1 \quad , \ldots, \quad 1 \end{bmatrix}$$

is singular. But $\bar{U}$ is not singular for regular systems, the proof being analogous to that of the preceding theorem.

q. e. d.

These two theorems emphasize again the erratic character of the movement of prices in function of the rate of profit in regular systems. Two values of $\tilde{p}(r)$ will never be proportionate at two different levels of the rate of profit, $n+1$ values of $\tilde{p}(r)$ will never be in the same n-dimensional hyperplane containing the origin in regular systems; n values of $\hat{p}(r)$ will be linearly independent and the $(n-1)$-dimensional hyperplane spanned by them will never contain any $(n+1)$st value of $\hat{p}(r)$ in regular systems (except when r is equal to an eigenvalue).

We have noted above that reswitching occurs if and only if there is a $(n+1)$-vector c such that $c\tilde{p}(r) = 0$ at two levels of the rate of profit, where c has to correspond to a feasible technique, i. e. $(a_0, l_0) = (a_1, l_1) - c \geqq 0$. We can now see that irregular systems are characterized by the existence of a vector c such that $c\tilde{p}(r) \equiv 0$. If c corresponds to a feasible technique, two techniques are compatible at all rates of profit. This is not really reswitching, but rather an indication of the odd and exceptional character of irregular systems: all points on the wage curve are "switchpoints" for (a_0, l_0) and (a_1, l_1). If we have the extreme case of an irregular system, i. e. if *prices equal values,* we find that *an alternative technique is compatible with the original technique either at all* *rates of profit or at most at one*. This can be seen from the fact that an alternative technique (a_0, l_0) will fulfill the equation

$$(1+r)\, a_0\, u + (1-r/R)\, l_0 = u_1$$

(where u is the vector of values and $(1-r/R)=w$ the wage rate) either identically or only for one rate of profit.

Reswitching in the sense that two techniques are equally profitable at two and only two rates of profit is therefore ruled out if prices are equal to values. We shall now show that the possibility of reswitching is characteristic for regular systems. However, whether it really takes place depends also on the alternative techniques which are available. Suppose that more than just two methods for the production of commodity one exist. If the most drastic neoclassical assumptions about technology are made, no reswitching can take place. For if we assume

1) constant returns to scale,

2) a technique for the production of one unit of commodity one is represented by a point in the non-negative orthant in R^{n+1} (where the first n components denote the amounts of required raw materials and the last the required amount of labour; the labour coefficient is always positive),

3) the feasibility of (a_0, l_0) implies the feasibility of all (a, l) where $(a, l) \geqq (a_0, l_0)$, and if we denote this $(n+1)$-dimensional technology set by TS and assume

4) strict convexity,

5) smoothness of the boundary of TS (the "technology frontier" BTS);

it is clear that a technique $(a_0, l_0) \in TS$ is eligible at a given rate of profit r, if $(a_0, l_0)\, \tilde{p}\,(r)$ is a minimum over all $(\bar{a}_0, \bar{l}_0) \in TS$. Eligible techniques are on the technology frontier BTS. The existence of a switchpoint (which is not an inner point of TS and therefore irrelevant) implies that $\tilde{p}\,(r)$ is orthogonal to some point c_0 on the (smooth) boundary $BT'S'$ of the set $\{x \in R^{n+1} | x = (a_1, l_1) - y, y \in TS\}$. Now the smoothness of $BT'S'$ insures that the normal is well defined in any point on the surface $BT'S'$ and since $\tilde{p}\,(r)$ never assumes twice the same value, *re*switching is impossible. Strict convexity insures, moreover, that there can be never more than one eligible technique corresponding to a given level of the rate of profit.

This reasoning looks persuasive and is effective in ruling out reswitching. However, it misses a very important point. A technique

(a_0, l_0) in the technology set TS which is eligible at any rate of profit is either on the boundary of the non-negative orthant $R_+{}^{n+1}$ or it is an inner point. But the latter is very unlikely, because no technique for the production of one commodity in an economy has ever been seen which used positive quantities of *all* basic commodities known in that economy as inputs. If strict convexity seems to suggest that an inner point of $R_+{}^{n+1}$ could ever be eligible as the most profitable technique for a given rate of profit, strict convexity is a dubious assumption. It is safe to assume that there will always be some zeros in the rows denoting the inputs of raw materials.

One might try to defend strict convexity by arguing somehow that a convex combination of the inputs to two different techniques for the production of the same commodity is technically superior to either of the techniques.

This argument may be justifiable for the combinations of some processes when the inputs to be combined are tools. (It is harder to find good examples when the inputs to be combined are raw materials, since the use value of a commodity frequently changes when the raw materials from which it is made are replaced by substitutes.) It is in fact possible to produce planks by means of a saw and by means of a hatchet, and perhaps advantageously with a combination of both.

But it is usually overlooked that strict convexity requires much more in the context of conventional neoclassical assumptions such as (1)—(3) above. For assumption (3) (free disposal) implies that a ton of steel — if it can be produced by means of a ton of coal and twenty man-hours — can also be produced by means of a ton of coal, twenty man-hours, 500 cherries and six elephants, since the latter two inputs may be "disposed of". Now there is no reason to assume that strict convexity obtains for a combination of these two "techniques" (in the same way as it obtained for a combination of saw and hatchet), for this would imply that a process of production could be made more productive by adding just any arbitrary input: the same amount of steel could be produced, using less coal and a few cherries more. Strict convexity implies, together with (3), that every input is a substitute for every other. Assumptions (3) and (4) are therefore not compatible in general. The assumption of a strictly convex $(n+1)$-dimensional technology set looks relatively innocent, since convexity is plausible, and so is the assumption of free disposal. Strict convexity then looks like an analytically useful additional hypothesis. But one should bear in mind that since no real process of production uses positive quantities of all commodities as inputs, the technology set is $(n+1)$-dimensional only because of assumption (3). For even if we admit assumptions (1) and (2) and suppose that TS is convex, we can hardly expect to be able to construct a feasible technique with all input coefficients positive, since we cannot expect to find for all i a technique $(\bar{a}_0, \bar{l}_0)$ which has a positive i-th component to every technique whose i-th input coefficient is zero. Hence we must conclude that TS would probably contain no positive point at all if we did not have assumption (3). The economically relevant techniques which do not contain disposable inputs are therefore all contained in the boundary of the non negative orthant R_+^{n+1}. They form a set which is less than $(n+1)$-dimensional. This point is never properly recognized despite the prevalence of zeroes in all empirical Leontief systems, because economists are used to think in terms of two or three sector models where a positive vector of inputs looks innocent.

It is nevertheless analytically convenient to retain free disposal (assumption (3)). There is no harm either in assuming the possibility of perfect substitutability for some groups of inputs, but general substitutability must not be assumed. Assumption (4) (strict convexity) is therefore to be replaced by the assumption of convexity. (The difference is fundamental, as we shall see.) Instead of (5), we introduce a new assumption: a technique $(a_0, l_0) \in TS$ will be said to contain no disposable inputs if none of the input coefficients of (a_0, l_0) can be reduced without increasing another. We assume that every technique without disposable inputs contains at least one coefficient which is zero. Our reasoning then implies that every point on the boundary of the technology TS, the technology frontier TS, should be assumed to be spanned by a set of points (a_0, l_0) where at least one of the components of each vector a_0 vanishes, or TS is obtained by adding disposable inputs to such points. The boundary of TS will therefore not be smooth where it intersects the boundary of R_+^{n+1}. Strict convexity may obtain on the boundary of R_+^{n+1} when two or even several inputs are substituted for each other, but cannot be expected to obtain with *all* substitutions and not for inner points of R_+^{n+1}.

Now it is important to note that an inner point of R_+^{n+1} can at best be eligible at any given rate of profit by a fluke. For $(a_0, l_0)\, \tilde{p}\,(r)$ can be a minimum for given r only if either (a_0, l_0) is on the boundary of R_+^{n+1}. Since no uniquely defined tangency plane of BTS exists in the boundary point, (a_0, l_0) will then in general remain eligible, except in fluke cases, if a small variation of the rate of profit takes place. Or (a_0, l_0) is an inner point of R_+^{n+1}. We abstract first from the existence of disposable inputs. In this case, (a_0, l_0) is a convex combination of at least two points (a_0', l_0') (a_0'', l_0'') which are on BTS and on the boundary of R_+^{n+1}, so that $(a_0, l_0)\,\tilde{p}\,(r)$ can be a minimum for given r only if $(a_0, l_0)\,\tilde{p}\,(r) = (a_0', l_0')\,\tilde{p}\,(r) = (a_0'', l_0'')\,\tilde{p}\,(r)$. Now any arbitrarily small variation of r in a *regular* system implies (because of theorem 4.1 above) that either $(a_0', l_0')\,\tilde{p}\,(r)$ or $(a_0'', l_0'')\,\tilde{p}\,(r)$ will become smaller than $(a_0, l_0)\,\tilde{p}\,(r)$. Hence (a_0, l_0) can be eligible at r only if (a_0, l_0) and (a_0'', l_0'') are switchpoints at r, hence only by a fluke. This explains why even if two techniques with some zero coefficients exist such that their linear combinations are positive, their joint use will not be observed.

But in an economy involving many commodities and processes it is likely (though we do not assume it) that not even groups of processes involving no disposable inputs will exist such that their convex combinations are positive. The boundary of the technology set BTS contains then positive points only because of the free disposal assumption. The positive points of BTS will therefore consist of pieces of hyperplanes which are parallel to at least one of the coordinate axes, and it follows that none of these points will ever be eligible in this case since $\tilde{p}\,(r)$ is positive.

These considerations may seem to imply that linear activity analysis provides a better representation of technology than the above set theoretical description, since linear activity analysis is based on the assumption of a finite number of constant returns to scale techniques. However, I do not want to exclude the possibility of continuous substitution altogether. Continuous or even differentiable substitution possibilities may obtain with pairs of groups of inputs. But if the technology frontier is not strictly convex everywhere, techniques on its boundary BTS will become eligible in discontinuous succession.

Thus we find that, as r varies in a regular system between zero and R, different techniques of BTS will become eligible. They will be on the boundary of R_+^{n+1}, or, if they are not, they are spanned by techniques which are eligible at the same rate of profit and which are on the boundary of R_+^{n+1}. In our discussion of the possibility of reswitching we may thus assume that the relevant eligible techniques are not inner points of R_+^{n+1}, and therefore that the technology frontier is not smooth in the neighbourhood of the relevant eligible points which are on the boundary of R_+^{n+1}.

As soon as edges and corners are admitted in the technology set, reswitching can easily occur in regular systems even if convexity is retained, for although the price vector will never assume the same value at two different levels of the rate of profit, its erratic behaviour may easily make it possible that the same corner will be profitable at two different levels of the rate of profit while another may be profitable in between.

Our results about the behaviour of the price vector and our discussion of the technology set will now allow us to give more precision to the statement that reswitching is "easily possible".

To begin with, we assume again that only one technique (a_0, l_0) for the production of one unit of commodity one exists which is an alternative to the actual technique (a_1, l_1). Suppose that the two techniques are different and equally profitable at $r = r_1$. How likely is it that a rate of profit $r_2 \neq r_1$, $0 \leqq r_2 < R$, can be found such that both techniques are equally profitable at $r = r_2$? We assume that the system is regular, for if prices are equal to values, reswitching is ruled out, and intermediate cases of irregular systems present uninteresting complications.

It is, of course, not possible to give an exact measure for the likelihood of reswitching in this case. But we can at least argue why reswitching is not just a mathematical fluke by considering the set

$$Y(r_1) = \{(a, l) \geqq 0 \mid (a, l)\, \tilde{p}(r_1) = (a_1, l_1)\, \tilde{p}(r_1)\}$$

of all "potential" techniques or vectors which are formally equally profitable as technique (a_1, l_1) at $r = r_1$ and the subset of $Y(r_1)$

$$Z(r_1) = \{(a, l) \in Y(r_1) \mid (a, l)\, \tilde{p}(r_2) = (a_1, l_1)\, \tilde{p}(r_2) \text{ for some } r_2 \neq r_1, 0 \leqq r_2 < R\}$$

which consists of all "potential" techniques or vectors which are as profitable as (a_1, l_1) at the given rate of profit r_1 and at *some* other rate of profit $r_2 \neq r_1$. That is to say, $Z(r_1)$ is the set of potential techniques which have one switchpoint with (a_1, l_1) at r_1, and another at some r_2 where r_2 is not the same for all points of $Z(r_1)$. Obviously, reswitching is a mathematical fluke if $Z(r_1)$ is only a "very small part" of $Y(r_1)$, for if the set of "potential" reswitchpoints $Z(r_1)$ is small in relation to the set of "potential" switchpoints $Y(r_1)$, the only actual alternative technique (a_0, l_0) — which is in $Y(r_1)$ — will only by a fluke be to be found in the set $Z(r_1)$. While I have not been able to construct an exact measure of $Z(r_1)$ in relation to some economic property of the system, one can at least prove:

Theorem 4.3:

The n-dimensional measure of $Z(r_1)$ as a percentage of the n-dimensional measure $Y(r_1)$ is positive, if the system is regular, and zero, if prices are equal to values.

Proof: If prices are equal to values u, prices are proportional to the vector of direct labour inputs (section 2 above), and "reswitching" of any potential technique (a_0, l_0) implies that (a_0, l_0) is compatible with (a_1, l_1) at all rates of profit (this section 4 above). Therefore $l_0 = l_1$, $a_0 u = a_1 u$ and $Z(r_1)$ is an $(n-1)$-dimensional set.

If the system is regular, the set

$$Y(r) = \{(a, l) \geqq 0 \mid [(a, l) - (a_1, l_1)]\, \tilde{p}(r) = 0\}$$

is an n-dimensional simplex in R_+^{n+1} spanned by its $n+1$ corner points on the coordinate axes of R_+^{n+1}. We have $(a_1, l_1) \in Y(r)$ for all r. Clearly

$$Z(r_1) = \bigcup_{\substack{0 \leqq r < R \\ r \neq r_1}} \{Y(r) \cap Y(r_1)\}$$

Because of Theorem 4.1, the simplices $Y(r)$ and $Y(r_1)$ have no corners in common except for a finite number of rates of profit r, $0 \leqq r < R$, $r \neq r_1$. Yet $Y(r) \cap Y(r_1)$ is not empty since $(a_1, l_1) \in$ $\in \{Y(r) \cap Y(r_1)\}$. (a_1, l_1) cannot be expected to be a positive vector. but $Y(r)$ and $Y(r_1)$ must have positive points in common.

To see this, denote the corners of $Y(r_1)$ by $\xi_i e_i$, $i = 1, \ldots, n+1$, (where e_i is the i-th unit vector) and the corners of $Y(r)$ by $\eta_i e_i$, $i = 1, \ldots, n+1$. (a_1, l_1) cannot be a corner of either $Y(r)$ or $Y(r_1)$ since the system is regular. Since $(a_1, l_1) \in \{Y(r) \cap Y(r_1)\}$ we can neither have $\xi_i < \eta_i$, $i = 1, \ldots, n+1$, nor $\xi_i > \eta_i$, $i = 1, \ldots, n+1$. Without loss of generality $\xi_i < \eta_i$, $i = 1, \ldots, t$, and $\xi_i > \eta_i$, $i = t+1, \ldots, n+1$; $1 \leqq t \leqq n$. The straight lines $\lambda \xi_i e_i + (1-\lambda) \xi_j e_j$ and $\lambda \eta_i e_i + (1-\lambda) \eta_j e_j$, $0 \leqq \lambda \leqq 1$, have one point $h_{i,j}$ in common for all pairs i, j with $1 \leqq i \leqq t$, $t+1 \leqq j \leqq n+1$. The $t(n+1-t)$ $(n+1)$-vectors h_{ij} and their convex combinations are in $Y(r) \cap Y(r_1)$, hence any convex combination with positive coefficients yields a positive point in $Y(r) \cap Y(r_1)$.

$Y(r) \cap Y(r_1)$ is therefore a $(n-1)$-dimensional set containing a positive point in R_+^{n+1}. We have to show that the correspondence $r \to \{Y(r) \cap Y(r_1)\}$, $0 \leqq r < R$, $r \neq r_1$, covers a n-dimensional subset of $Y(r_1)$ containing an open n-dimensional set.

To prove this we note that the points on $Y(r_1)$ covered by the correspondence $r \to \{Y(r) \cap Y(r_1)\}$ are for sufficiently small variations of r points which are also covered by the mapping

$$\phi\colon (r, \varrho_2, \ldots, \varrho_n) \to R_+^{n+1}$$

given by

$$(a, l) = \{(a_1, l_1)\, \tilde{p}(r),\ (a_1, l_1)\, \tilde{p}(r_1),\ \varrho_2, \ldots, \varrho_n\} \{M(r)\}^{-1}$$

where

$$M(r) = \{\tilde{p}(r), \tilde{p}(r_1), \tilde{p}(r_2), \ldots, \tilde{p}(r_n)\}$$

with $r, r_1, \ldots, r_n$ all different, $0 \leqq r_i < R$, and where $\varrho_2, \ldots, \varrho_n$ are parameters, varying between zero and $+\infty$. Conversely: if $(a\ l) = \phi(r, \varrho_2, \ldots, \varrho_n)$ and if $(a, l) \geqq 0$, we have $(a, l) \in \{Y(r) \cap Y(r_1)\}$. There is a point $(\bar{a}, \bar{l}) \in Y(r_1) \cap Y(r)$, $r \neq r_1, \ldots, r_n$, such that $(\bar{a}, \bar{l}) > 0$. With $(\bar{a}, \bar{l})\ \tilde{p}(r_i) = \bar{\varrho}_i > 0$, $i = 2, \ldots, n$, the mapping ϕ maps the point $(r, \bar{\varrho}_2, \ldots, \bar{\varrho}_n) \in R^n$ onto the positive point $(\bar{a}, \bar{l})$ on $Y(r_1)$. Since the image point of ϕ is positive and since ϕ is continuous and one-to-one[9] in a sufficiently small n-dimensional neighbourhood of $(r, \bar{\varrho}_2, \ldots, \bar{\varrho}_n)$, the correspondence $r \to Y(r) \cap Y(r_1)$ covers an open n-dimensional set in $Y(r_1)$ for small variations of r.

q. e. d.

The geometry of $Z(r_1)$ increases in complexity with n, i. e. with the number of commodities. If $n = 2$, $Y(r)$ is a two dimensional simplex in $R_+{}^3$ and $Z(r)$ can be drawn. The triangle in Fig. 1 represents $Y(r_1)$, the shaded area $Z(r_1)$.

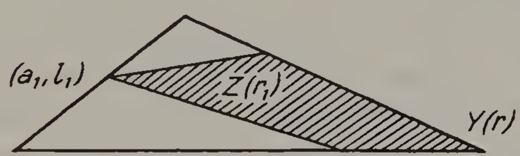


Fig. 1. The triangle represents the set of potential techniques which are as profitable as the actual technique (a_1, l_1) at rate of profit r_1 and $Z(r_1)$ represents the set of potential techniques which are as profitable as (a_1, l_1) also at some other rate of profit

$Z(r_1)$ degenerates to a straight line if prices are equal to values. The area of $Z(r_1)$ is the greater the more directions in space are

[9] There are in fact exceptional points. If the mapping is not one to one in $(\bar{a}, \bar{l})$, it means that ϕ maps some point $(r', \varrho_2', \ldots, \varrho_n') \neq (r, \varrho_2, \ldots, \varrho_n)$ onto $(\bar{a}, \bar{l})$. It follows at once that $\bar{\varrho}_i = \varrho_i'$, $i = 2, \ldots, n$, since only the first column of matrix $M(r)$ varies with r. Therefore, since $(\bar{a}, \bar{l})\ \tilde{p}(r) = (a, l_1)\ \tilde{p}(r)$, $(a, \bar{\bar{l}})\ \tilde{p}(r') = (a_1, l_1)\ \tilde{p}(r')$, and $(\bar{a}, \bar{l})$ turns out to be a switchpoint with (a_1, l_1) not noly at r and r_1, but also at r'. One has therefore to choose $(\bar{a}, \bar{l})$ such that it does not happen to be in the $(n-2)$ dimensional subset $Y(r) \cap Y(r_1) \cap Y(r')$ of $Y(r) \cap Y(r_1)$ for any r' in a sufficiently small neighbourhood of r. This will always be possible since $Y(r) \cap Y(r_1) \cap Y(r')$ does not cover more than a small part of $Y(r) \cap Y(r_1)$ if r' varies only a little. All these arguments depend crucially on the regularity of the system.

assumed by the vector $\tilde{p}(r)$, $0 \leqq r < R$. $Z(r_1)$ can never cover the whole of $Y(r_1)$, however. No point on the $(n+1)$ $-st$ coordinate axis belongs to $Z(r)$ since the equation $(0, \bar{l})\ \tilde{p}(r) = (a_1, l_1)\ \tilde{p}(r)$, i. e. the equation $\bar{l} = \hat{p}_1(r)$ is fulfilled for at most one rate of profit.

The larger the area $Z(r_1)$ of potential techniques which lead to reswitching, the greater the likelihood that the only actual alternative technique $(a_0, l_0) \in Y(r_1)$ will be in $Z(r_1)$.

So far, we have assumed that only one actual alternative technique (a_0, l_0) was available. More precisely, the technology frontier was spanned by (a_1, l_1) and (a_0, l_0). r_1 was the rate of profit at which (a_0, l_0) was as profitable as the original technique (a_1, l_1). In order to determine wether (a_0, l_0) was likely to lead to reswitching we looked at the set $Z(r_1)$ of all potential techniques which are alternatives to (a_1, l_1) at $r = r_1$ and at some other rate of profit. Since the set $Z(r_1)$ was found to be of the same dimension as the set $Y(r_1)$, the likelihood of (a_0, l_0) being in $Z(r_1)$ was not negligible. No theorem could be proposed expressing the measure of $Z(r_1)$ as a percentage of the measure of $Y(r_1)$ in function of some economic property of the system; all that could be said for sure was that the measure of $Z(r_1)$ as a percentage of the measure of $Y(r_1)$, hence the likelihood of (a_0, l_0) being in $Z(r_1)$, was not zero.

This remains true, if the technology frontier is spanned by more than two points, although we get a complication.

An alternative technique (a_0, l_0) which is as profitable as (a_1, l_1) at r_1 with both techniques being eligible at r_1, may not be eligible any more at some other rate of profit $r_2 \neq r_1$ at which (a_0, l_0) is as profitable as (a_1, l_1). There will then be reswitching in that the two techniques are equally profitable at r_1 and at r_2, but they are eligible only at r_1. Both are less profitable at r_2 than some third technique. By making sufficiently bold assumptions about the technology frontier one can ensure that whenever two techniques are equally profitable at two different rates of profit, there will always be a third technique dominating them by being more profitable than either at one of the two rates of profit. In the extreme case, if strict convexity and smoothness of the frontier are assumed, one gets rid of reswitching in so far as there is then only one eligible technique at each rate of profit.

But I have tried to show that strict convexity and smoothness are highly dubious assumptions. If corners of the technology frontier are admitted, it may still be that whenever two corners are equally profitable at two rates of profit, a third will be eligible at one of these two rates of profit. But there is no reason why this should be so in general. If (a_0, l_0) is as profitable as (a_1, l_1) at $r=r_1$, both being eligible, there is a positive possibility that (a_0, l_0) will be in $Z(r_1)$, i. e. as profitable as (a_1, l_1) at some $r=r_2$. And if (a_0, l_0) is in $Z(r_1)$, there is again surely a positive possibility that (a_0, l_0) is not dominated by a third technique at $r=r_2$. The fact that the product of two probabilities may be a probability smaller than either does not reduce a possibility to a fluke.

5. Wicksell Effects

We may try to pursue the pure logic of a Sraffa system a little further by applying the formula derived above for systems with simple roots to the analysis of Wicksell effects. This will be all the more useful since perverse movements of the capital labour ratio are at least as relevant for the criticism of neoclassical theory as reswitching, and in discussing them we have the advantage of not having to make any assumption about alternative techniques, be it in the conventional form of Neoclassical assumptions about a strictly convex technology set or, more cautiously but still hypothetically, a book of blue prints[10].

We calculate the capital labour ratio of a stationary non basic system where the non-basics are all pure consumption goods, *viz.* they are produced by means of labour and basics alone. The basic part of the system is assumed to be regular with no multiple roots in the characteristic equation. The system is supposed to produce a surplus of non-basics only, and the basket of non-basics in the surplus will be taken as the numéraire for prices. The model represents the obvious generalisation of the conventional two-sector system with one basic commodity and one non-basic serving as "numéraire". Formally this may be expressed as follows:

The input matrix is given by

$$A=\begin{bmatrix} A_1^1 & 0 \\ A_2^1 & 0 \end{bmatrix}$$

where A_1^1 is a (n, n) indecomposable matrix for the basic part of

[10] The book of blue prints may always contain goods which may become commodities in the new system if the new technique is adopted while they were not commodities in the old system. This is an awkward possibility since it implies that we must be able to list and measure goods which have not been listed or measured or even been defined as separate goods by the market. There is no such methodological difficulty involved when we calculate what prices of production of *given* commodities would be if the rate of profit changed with techniques remaining unchanged.

the system and A_2^1 a (m, n) matrix. The output matrix (unit matrix) I and the labour vector l are partitioned accordingly. The net surplus of consumption goods to be produced is given by a $(n+m)$-row vector $d=(d_1, d_2)$ where the n-vector d_1 equals zero and where the m-vector $d_2>0$. Activity levels q are then given by

$$q = d\,(I-A)^{-1} = [d_2 A_2^1\,(I_1^1 - A_1^1)^{-1}, d_2].$$

Total labour in the economy is taken to be unity, i. e. $ql=1$. The price equations are

$$p_1 = (1+r)\,A_1^1 p_1 + w l_1$$
$$p_2 = (1+r)\,A_2^1 p_1 + w l_2$$

where $dp \equiv 1$ so that

$$w\,(r) = \frac{1}{d_2 \hat{p}_2\,(r)}\,;\ \hat{p} = \frac{p}{w}.$$

The capital labour ratio is

$$k = \frac{K}{L} = \frac{q\,Ap}{w\,q\,l} = \frac{q\,A^1 \hat{p}_1}{d_2 \hat{p}_2}.$$

The n eigenvectors of A_1^1 are linearly independent so that the sum of the inputs of basics in the processes of non basics can be represented as a linear combination of them

$$d_2 A_2^1 = \sum_{i=1}^{n} \lambda_i q_i$$

where

$$(1+R_i)\,q_i A_1^1 = q_i,$$

and where we normalize the q_i to $q_i l_1 = \frac{1}{1+R_i}$ (since the system is regular, we have $q_i l_1 \neq 0$).

Using

$$q_i\,[I - (1+r)\,A] = q_i \left(1 - \frac{1+r}{1+R_i}\right)$$

the formula for the capital labour ratio may be simplified

$$k = \frac{d_2 A_2^1\,(I_1^1 - A_1^1)^{-1} A_1^1 \hat{p}_1 + d_2 A_2^1 \hat{p}_1}{d_2\,(1+r)\,A_2^1 \hat{p}_1 + d_2 l_2}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i \left(\frac{1+R_i}{R_i}\,\frac{1}{1+R_i}\,\frac{1+R_i}{R_i-r} + \frac{1+R_i}{R_i-r}\right) q_i l_1}{d_2 l_2 + (1+r) \sum_{i=1}^{n} \lambda_i \frac{1+R_i}{R_i-r}\,q_i l}$$

so that it is shown to depend essentially on the "eigenvalues" R_i and the coefficients λ_i by means of which the inputs of basics to the processes of non basics are expressed as a linear combination of the eigenvectors:

$$k = \frac{\sum_{i=1}^{n} \lambda_i \frac{1+R_i}{R_i} \frac{1}{R_i - r}}{d_2 l_2 + \sum_{i=1}^{n} \lambda_i \frac{1+r}{R_i - r}}.$$

The capital labour ratio is thus represented as a rational function of r in explicit form. It reduces to a constant in essentially only one case: if by coincidence $d_2 A_2^1 = \lambda_1 q_1$ and, also by coincidence, $d_2 l_2 = \lambda_1$ we get $k = 1/R_1$. The same simple result is obtained in a basic Sraffa system (where the surplus consists of basics only) in standard proportions and also in a basic Sraffa system where prices are equal to values. Here, where non-basics are involved, the situation is more complicated, but one can easily show that $d_2 l_2 = \lambda_1$ implies that the organic composition is the same in both sectors of a two sector model where the first sector produces a basic good by means of itself and labour and where the second sector produces a non-basic by means of the output of the basic sector and labour.

If the vector of inputs of basics to non-basic industries does not happen to be proportional to the standard commodity of the basic part of the system and if the coefficient of proportionality does not happen to be equal to total labour employed in non-basic industries, the capital labour ratio may vary in almost any conceivable way with the rate of profit. The point is that these variations are due to the structure of the basic part of the system, for the formula shows that the capital labour ratio of the entire system depends crucially on the eigenvalues and the eigenvectors of the basic part of the system. This result confirms the thesis the Wicksell effects are mainly due to the interaction of the basic industries. It is therefore out of place to discuss, as is often done, reswitching or Wicksell effects in terms of two sector models with one basic and one consumption good, for the relevant problems of capital theory are visible only in models involving several basic goods.

6. The Capital-Wages Ratio

The conclusion of the previous section is, when separated from the argument supporting it, not very impressive. To say that reswitching may take place or may not take place, or that the capital labour ratio may move in either direction except if it is by coincidence constant is very nearly an empty statement which can be important only as a warning to those who are still trying to get round the criticisms made against neoclassical theory by means of some clever and artificial construction. Recently Kazuo Sato[11] has claimed in the Quarterly Journal of Economics that "the neoclassical postulate ... remains one of the most powerful theorems in economic theory" (p. 355). He supports this claim by enlarging on Professor Samuelson's construction of a surrogate production function. He takes Samuelson's old two sector model[12] without assuming as Samuelson did that the organic compositions in both sectors are the same. He is nevertheless able to show that reswitching will not occur provided sufficiently bold assumptions about available techniques are made, i. e. provided the existence of a "technology frontier" is assumed and provided the substitution properties of the technology frontier are appropriate. His article is a very nice exercise in the analysis of two sector models with variable techniques, but I hope to have reminded the reader (as Sato himself is sufficiently candid to admit) that the real difficulties of capital theory begin when we are dealing with a many sector model, i. e. essentially when we are dealing with a model involving several *basic* goods. Our model provides a critique of Sato, since it is a direct generalisation of Sato's version of Samuelson's two sector model, in that basics are here the only inputs to production besides labour and in that the non-basics furnish the standard of prices.

This article has confirmed that prices of production follow a twisted curve in function of the rate of profit in regular systems involving several basic industries. The consequent complicated movement of prices is excluded only if prices are equal to values. It is "evened out" for the standard commodity. In the general case it is such that reswitching becomes an irrefutable possibility if it is recognized that the technology frontier is likely to have corners. And even if the technology frontier is assumed to be smooth, there will still be Wicksell effects for a *given* technique. It is therefore no wonder when people complain that the reswitching controversy has made capital theory awfully difficult. However, I want to conclude with a more constructive remark. The difficulties with the

---

[11] Kazuo Sato; op. cit.

[12] P. A. Samuelson: Parable and Realism in Capital Theory: the Surrogate Production Function, Review of Economic Studies 39 (1962), pp. 192—206.

capital labour ratio are in part due to the fact that it is a hybrid concept in that the measurement of capital requires a measurement in terms of absolute prices while labour is measured in terms of physical units. If instead of the capital labour ratio we use the (perhaps pseudo-marxist) concept of an "organic composition of capital expressed in price terms" we get rid of the problem of choosing an appropriate standard of prices. The "organic composition of capital in price terms" is simply the relation of total capital to total wages in the economy: K/W. Given the technique and the labour force this expression depends on the rate of profit. But it does not depend on the chosen standard of prices because the price standard occurs both in the numerator and the denominator. Moreover, the capital-wages ratio is at least a monotonic function of the rate of profit in a single product system. In formulas:

$$\frac{K}{W} = \frac{qAp}{wql} = \frac{qA\hat{p}}{ql}$$

where q denotes activity levels, p, w prices in terms of any standard and $\hat{p}$ prices in terms of the wage rate. Since prices in terms of the wage rate rise monotonically with the rate of profit for a given technique, the capital wages ratio will do the same (between zero and the maximum rate of profit).

The capital-wages ratio is the relevant concept from the microeconomic point of view when the entrepreneur wishes to assess the relative *cost* of capital and labour; when he wants to compare "capital" and labour in physical terms he has to compare machines, raw materials and men. The concept of the capital-wages ratio is equally useful in macroeconomics since it relates the distribution of income between profits and wages P/W with the rate of profit P/K:

$$r = \frac{P}{K} = \frac{P/W}{K/W}.$$

If the curve indicating the capital-wages ratio in function of the rate of profit for a given technique (the "capital-wages function") shifts upwards or downwards because of a technological change (technical progress), it follows that the rate of profit is lowered or raised accordingly if the distribution of income (P/W) is fixed. This conclusion which is important for any discussion of the interdependence of income distribution, technical progress and the rate of profit may be drawn because the capital-wages function is (for a given technique) a monotonically rising function of the rate of profit, and because it is, moreover, a pure number, i. e. independent of the monetary standard of prices[13] (see Fig. 2).

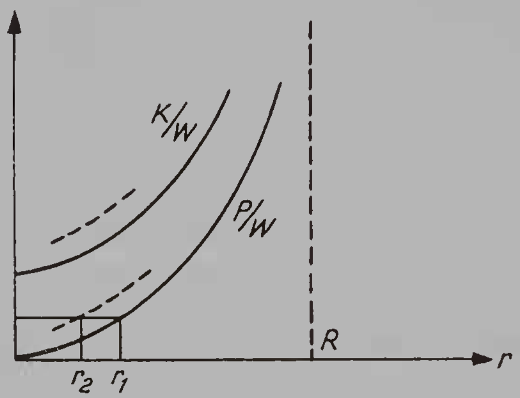


Fig. 2. The capital-wages ratio and the profits-wages ratio as functions of the rate of profit (capital-wages function and profit-wages function respectively). If the capital-wages function shifts upwards because of technological change, the profit-wages function does the same. The shifting of the curves (dotted lines) entails a fall in the rate of profit from r_1 to r_2 if the actual profit wages ratio in the economy is not affected by the technological change

I believe that J. Robinson and N. Kaldor were right in asserting that the dilemma posed by the heritage of neoclassical theory can only be overcome by shifting attention from processes of substitution to technical innovation. A discussion of the macro effects of technical progress involves an analysis of the relation between "microeconomic" switches of technique in physical terms and macroeconomic changes of "factor ratios".

If such an analysis makes it possible to express the effect of "microeconomic" changes of techniques in terms of shifts of the macroeconomic capital-wages function we should get nearer to a postneoclassical theory of the interaction between progress, distribution and profitability.

Address of author: Prof. Dr. Bertram Schefold, Fachbereich Wirtschaftswissenschaften, Johann-Wolfgang-Goethe-Universität, D-6000 Frankfurt, Federal Republic of Germany.

[13] This is discussed in detail in B. Schefold: Fixed Capital as a Joint Product and the Analysis of Accumulation with Different Forms of Technical Progress; to be published.

[20]

RETURNS TO SCALE AND THE SWITCH IN METHODS OF PRODUCTION

*Ian Steedman**

Many words have been written on the question whether Sraffa's analysis in his *Production of commodities by means of commodities* does, or does not, require an assumption of constant returns to scale. (See space amongst many other examples - Burmeister/2a/, /2b/; Eatwell/3/; Levine /5a/, /5b/, /5c/; Pasinetti /6, p.92/; Roncaglia /7, pp. 22-6, 106-7, 117-8/). Yet, curiously enough, it is seldom mentioned that such an assumption is quite definitely required in Part III of Sraffa's book, devoted to the analysis of switches in the methods of production as distribution changes. This note is directly solely to establishing the fact just referred to and no comment will be made on any other aspect of the matter beyond the (unargued) assertion that returns questions are unambiguously irrelevant to Parts I and II of Sraffa's book.

Sraffa's Preface

One must naturally turn first to Sraffa's well-known Preface, for it is there that Sraffa refers to the question of constant returns. Since the relevant passage is often quoted *incompletely*, it is reproduced here in full:

"Anyone accustomed to think in terms of the equilibrium of demand and supply may be inclined, on

* I should like to thank J.Eatwell, P.Garegnani, L.L.Pasinetti and N.Salvadori for their comments on an earlier version of this note.

6

> reading these pages, to suppose that the argument rests on a tacit assumption of constant returns in all industries. If such a supposition is found helpful, there is no harm in the reader's adopting it as a temporary working hypothesis. In fact, however, no such assumption is made. No changes in output and (at any rate in Parts I and II) no changes in the proportions in which different means of production are used by an industry are considered, so that no question arises as to the variation or constancy of returns. The investigation is concerned exclusively with such properties of an economic system as do not depend on changes in the scale of production or in the proportions of 'factors'". /9, p. v/.

The firm tone of most of this passage - e.g. "*no* such assumption is made" and "the investigation is concerned *exclusively* with such properties..." (my emphases) - has led many commentators to consider the matter closed. And they may have been encouraged so to consider it by the entry in Sraffa's index which reads bluntly "Constant returns, not assumed v". A firm tone, however, is never a justification for careless reading and the parenthesis "(at any rate in Parts I and II)" should have led rather more commentators to wonder whether returns questions are irrelevant to Sraffa's Part III and indeed to wonder whether it is really true that "No changes in output" are relevant in that Part.

Outputs must change when methods are switched

To see that, other than by a fluke, a switch in production methods must involve a change in gross outputs, it will suffice to consider a simple two commodity economy[1]. (It is, of course, the constancy or otherwise of *gross* outputs that matters for returns questions). We proceed by supposing that gross output changes are *not* involved in a switch and seeing how restrictive are the conditions

7

Although the fourth sentence of Sraffa's Preface does indeed state that *no* changes in outputs are considered, it could be objected to the above argument that only the constancy of X_1 is relevant to returns questions. Suppose then that X_2' can differ from X_2; (1), (2) and (3) above are unaffected but (4) must now be replaced by

$$X_2' = X_{21} + X_{22}' + Y_2'. \tag{7}$$

It follows that (5) is unaffected but that (6) must be replaced by - from (2) and (7) -

$$X_2 - X_{22} - Y_2 = X_2' - X_{22}' + Y_2'. \tag{8}$$

For given values of Y_1 and Y_2, there is still no reason at all why there should exist an X_2' such that (5) and (8) yield non-negative values for both Y_1' and Y_2'. Thus even the assumption that X_1 *alone* is independent of the method used would be completely arbitrary in the analysis of method switches.

It has been assumed so far that both commodities are "basics", whichever method is used for commodity 2. If, by contrast, commodity 2 were "non-basic" in one or in both methods, so that X_{21} and/or $X_{21}' = 0$, this would make no difference at all to the above argument, since neither X_{21} nor X_{21}' appears in (5), (6) or (8). Again, if commodity 1 were "non-basic" with both methods, so that $X_{12} = X_{12}' = 0$, then (5) would readily be satisfied but it would still be quite arbitrary to suppose that (6) and (8) would always yield non-negative net outputs. (If commodity 1 were "basic" with one method but "non-basic" with the other, (5) would again impose a significant restriction).

It may be concluded that if the analysis of switches in methods of production is not to be restricted to ridiculously special cases, it *must* allow for changes in gross outputs[2]. It therefore has to be considered whether Sraffa's analysis in his Part III presupposes constant returns.

8

under which that is possible. Suppose then that there is one available method for the production of commodity 1 but that there are two available for the production of commodity 2. When the first alternative method is used in industry 2, let the commodity flows be given by

$$X_1 = X_{11} + X_{12} + Y_1 \tag{1}$$

$$X_2 = X_{21} + X_{22} + Y_2, \tag{2}$$

where X_i (Y_i) is gross (net) output of commodity i and X_{ij} is the quantity of i used as an input in industry j. (All the X_{ij} are taken to be positive, for the moment, so that both commodities are "basics"). When the second method is used in industry 2, let the commodity flows be given by

$$X_1 = X_{11} + X'_{12} + Y'_1 \tag{3}$$

$$X_2 = X_{21} + X'_{22} + Y'_2, \tag{4}$$

where the dash notation is obvious and, it will be noted, X_1, X_2, X_{11} and X_{21} are the same as in (1) and (2) because, by assumption, *no* gross output changes with the switch in method.

From (1) and (3),

$$X_{12} + Y_1 = X'_{12} + Y'_1, \tag{5}$$

while from (2) and (4),

$$X_{22} + Y_2 = X'_{22} + Y'_2 \tag{6}$$

Now for given values of Y_1 and Y_2 (Y'_1 and Y'_2) there is absolutely no reason why (5) and (6) should yield non-negative values of both Y'_1 and Y'_2 (Y_1 and Y_2). Thus the assumption that both X_1 and X_2 are independent of the method in use is quite arbitrary - the analysis of switches in method *cannot* reasonably be based on the assumption that *no* gross output changes.

9

A method switch for a non-basic commodity

We may follow Sraffa's example /9, §92/ by considering first the case of a method switch for a *non-basic* commodity. It has been seen above that, in general, such a method switch will involve a change in the gross output of *at least* one basic industry. (For example, *net* output of pig-iron may be zero and method 1 for the non-basic may use no pig-iron as an input, while method 2 for the non-basic does so use pig-iron. Thus pig-iron gross output will depend on the non-basic method used - and so, in general, will *other* basic gross outputs as a result.) We must therefore ask whether constant returns must be assumed in basics production. Let non-basic method i use the vector $\underline{\alpha}_i$ of produced (basic) inputs and β_i labour to produce 1 unit of non-basic (i=1,2). Let the *average* input-output ratios in the basic sector, when method i is used for the non-basic, be shown by the matrix A_i for produced inputs and by the vector $\underline{b}_i$ for labour (i = 1,2). If there are n basics, let $\underline{p}$ be the (1 x n) vector of basics prices and π be the price of the non-basic.

At a switch point between the two non-basic methods we have:

$$(1+r)\,\underline{p}\,A_1 + w\,\underline{b}_1 = \underline{p} \tag{9}$$

$$(1+r)\,\underline{p}\,\underline{\alpha}_1 + w\,\beta_1 = \pi \tag{10}$$

$$(1+r)\,\underline{p}\,A_2 + w\,\underline{b}_2 = \underline{p} \tag{11}$$

$$(1+r)\,\underline{p}\,\underline{\alpha}_2 + w\,\beta_2 = \pi. \tag{12}$$

Also, since Sraffa suggests any arbitrary commodity as standard of value for this analysis /9, p.104/, let us take as the standard the bundle of basics shown by the vector $\underline{z}$, so that

$$\underline{p} \cdot \underline{z} = 1. \tag{13}$$

10

Now (9)-(13) provide (2n+3) equations in just $(n + 3)$ unknowns - thus, flukes apart, (9)-(13) are overdetermined and do *not* yield the profit rate, wage rate and prices corresponding to a switch-point. The only (non-fluke) exception to this result is the case of *constant returns to scale in basics production*, for in this case $A_1 = A_2$, $\underline{b}_1 = \underline{b}_2$ and (11) merely duplicates (9), so that we have only $(n + 3)$ independent equations in the $(n + 3)$ unknowns.

It may be of interest now to repeat the argument in terms closer to those used by Sraffa in his §92. From (9), (10) and (13) we can, in general, express π as a function of r for method 1; from (11), (12) and (13) we can, in general, express π as a function of r for method 2. Sraffa's argument is then that, at any given r, that method will be chosen which has the lower value of π. A switch-point occurs when both methods have the same π for a given r. But, without $A_1 = A_2$ and $\underline{b}_1 = \underline{b}_2$, such a "switch-point" is only a *pseudo* switch-point, since the two methods will give *different* values of w and of $\underline{p}$. And indeed, away from a pseudo switch point, the method yielding the lower π might also yield the *lower* wage *measured in terms of* whichever composite commodity the real wage consists of (which will not generally be $\underline{z}$, of course).

It may be concluded that Sraffa's analysis of switches in production methods for a non-basic implicitly assumes that *every* basic industry has *either* a constant gross output *or* constant returns to scale: the only non-arbitrary interpretation of this implicit assumption is that *every* basic industry exhibits constant returns[3].

A method switch for a basic commodity

Consider now a system of just n basic commodities; let there be one available method for each of the industries 1 to (n-1) but two available methods for industry n. Let A_i and $\underline{b}_i$ represent the *average* input-output ratios, for basics and labour respectively, when method i is in

11

use for commodity n (i = 1,2). Equations (9), (11) and (13) above can now be re-interpreted to refer to the present case: they provide (2n + 1) equations in (n + 2) unknowns and are thus generally overdetermined (I assume $n \geq 2$!). Of course, if each of industries 1 to (n-1) has constant output or constant returns then (9) and (11) have (n-1) equations in common, so that there are only $[(n-1) + 3] = (n+2)$ independent equations and hence we have a determinate switch point[4]. But if j of the industries 1 to (n-1) change their *average* coefficients with a method switch, then there are $[(n-j-1) + 2j + 3] = (n+j+2)$ independent equations in the (n+2) unknowns: if even one of the industries in question exhibits neither constant output nor constant returns (j=1) then (9), (11) and (13) are overdetermined[5]. We again find that the only non-arbitrary basis for the usual analysis of switches in production methods is the assumption of constant returns in *every* non-switching industry.

A query

I should like to conclude by posing - but not answering - the obvious question raised by the above argument: "How should the analysis of the choice of production methods be carried out when there are *not* constant returns in every non-switching industry?"

In the now-conventional analysis of method choice (with constant returns and single-product industries) one draws the real wage rate/profit rate frontier for each method and then asserts that at any given real wage rate (profit rate) that method will be used which yields the highest profit rate (real wage rate). This argument turns crucially on two properties of such systems: first, that it makes no difference which measure of the real wage rate is used and, secondly, that the method on the outer frontier will be the *cheapest* method of production, not only evaluated in its "own" price system but also evaluated in the price systems of other methods. (See, e.g., Garegnani /4, especially the first footnote to the second section/ and Pasinetti /6, Capitolo VI/). Now it is implicit in the arguments above that, with *non*-constant returns, neither of these properties is ensured, even for *given* (but different) output levels for the alternative production systems[6]. This would seem to raise large questions about the operation of the competitive process,as well as the "technical" question of analysis posed above, but I shall refrain from speculating on such matters here.

(Pervenuto il 28 febbraio 1980)

Notes

1. The reader will easily extend the argument to the n commodity case - and, indeed, to that of joint-production, noting, in the latter case, that constant activity levels - and not merely constant gross outputs of commodities - are needed to avoid returns questions. (Only single-product systems will be considered in this paper).

2. Of course, it would always be *possible* to compare two systems with *given* gross outputs, even when it followed that one of them yielded some *negative* net outputs. But of what interest would such a comparison be (in a closed economy)? With *non*-constant returns to scale, such a comparison might, in any case, show each technique to be more profitable at its (or at the other's) gross output pattern! What would then be deduced about the choice of technique?

3. It may also be noted that in his fig. 7 /9, p.104/ Sraffa takes the maximum possible profit rate to be independent of the method used for the non-basic. This again implies that *every* basic industry has either constant output or constant returns. If $A_1 \neq A_2$ then their Perron-Frobenius roots (and hence the corresponding maximum profit rates) will be equal only by a complete fluke.
4. Note that the third paragraph of Sraffa's §93 quite obviously assumes that every non-switching industry has constant output or constant returns (only the latter being plausible).

5. As has been pointed out to me by Neri Salvadori (Università di Napoli), even if a non-negative solution should (by a fluke) exist for such a case, one should not rush to assume that it has the usual *economic* meaning. It would do so only if, in the neighbourhood of such a "switch-point", there were always a production system which unambiguously minimized costs. C.f., Bharadwaj /1, esp. pp. 412, 416/.

6. It is interesting to note that the choice of method problem for the

13

joint-production case also encounters the difficulty that there may not be a method which minimizes costs in *all* price systems (even with constant returns). See Salvadori /8/.

References

/1/ Bharadwaj, K. "On the Maximum Number of Switches Between Two Production Systems", *Schweizerische Zeitschrift für Volkswirtschaft und Statistik,* 1970, pp. 409-29.

/2a/ Burmeister, E. "A Comment on 'This Age of Leontief... and Who?'", *Journal of Economic Literature,* 1975, pp. 454-7.

/2b/ - "The Irrelevance of Sraffa's Analysis without Constant Returns to Scale", *Journal of Economic Literature,* 1977, pp. 68-70.

/3/ Eatwell, J. "The Irrelevance of Returns to Scale in Sraffa's Analysis", *Journal of Economic Literature,* 1977, pp. 61-8.

/4/ Garegnani, P. "Heterogeneous Capital, the Production Function and the Theory of Distribution", *Review of Economic Studies,* 1970, pp. 407-36.

/5a/ Levine, A.L. "This Age of Leontief... and Who? An Interpretation", *Journal of Economic Literature,* 1974; pp. 872-81.

/5b/ - "This Age of Leontief... and Who?' A Reply", *Journal of Economic Literature,* 1975, pp. 457-61.

/5c/ - "The Irrelevance of Returns to Scale in Sraffa's Analysis: A Comment", *Journal of Economic Literature,* 1977, pp. 70-2.

/6/ Pasinetti, L.L. *Lezioni di teoria della produzione,* Bologna, Il Mulino, 1975.

/7/ Roncaglia, A. *Sraffa e la teoria dei prezzi,* Bari, Laterza, 1975.

/8/ Salvadori, N. "Mutamento dei metodi di produzione e produzione congiunta. Un commento al §96 di *Produzione di merci a mezzo di merci", Studi Economici,* 1979, pp. 79-94.

/9/ Sraffa, P. *Produzione di merci a mezzo di merci,* Torino, Einaudi, 1960.

[21]

On Non-Basic Commodities*

By Enrico Zaghini

1.

In a very interesting article published in this review[1] Newman discusses the first part of Sraffa's book[2] regarding simple production. Referring to the theory of non-negative matrices, he proves that all the main propositions contained in this part are valid. The only point in which Newman differs from Sraffa is that regarding the existence or non-existence of non-basic commodities[3]. In fact Newman presents a system constituted by a basic and a non-basic industry, in which it is not possible to determine two positive prices and states that the conditions that should be satisfied by matrices containing non-basics in order to have positive prices seem not to have economic meaning. Therefore unless we are "to confine ourselves to a rather odd and restricted class of situations ... the economic rational for which seems obscure", Newman concludes that "either we must abandon one of Sraffa's Assumptions, or we must assume that 'non-basics' do not exist"[4]. He chooses the second alternative and carries on his analysis on the assumption that non-basics do not exist.

At this point my opinion diverges radically from that of Newman's, as it seems to me that his assumption is far more restrictive than the one he seeks to avoid. In fact it certainly implies the exclusion of a releyant number of commodities which cannot be ignored. When no Ricardian hypothesis is made of a natural wage-rate represented by a fixed basket of goods, in the category of non-basics there enters besides the luxury goods, strictly speaking, also the great majority of consumer goods. Taking this into account I cannot absolutely agree with Newman when he states that non-basic commodities "are of greater mathematical interest than economic"[5]. Newman's attitude is, however, comprehensible because the exclusion of non-basics, which from the mathematical point of view can be expressed by saying that the technology matrix must be

* This is part of a paper presented in May 1964 at the Institute of Political Economics, Faculty of Statistics, Rome University.

[1] Peter *Newman*, Production of Commodities by Means of Commodities, Schweizerische Zeitschrift für Volkswirtschaft und Statistik, 1962, n. 1.

[2] Piero *Sraffa*, Production of Commodities by Means of Commodities, Cambridge University Press, 1960.

[3] Basic commodities are those commodities which enter directly or indirectly into the production of all commodities. Non-basic commodities, which are also called luxury products, are all others.

[4] *Newman*, op. cit., p. 67.

[5] *Newman*, op. cit., p. 75.

indecomposable, is normally made by the quasi-totality of economists which deal with multi-sectoral models. But the reason why it is made does not depend on its economic meaning, but simply on the circumstance that it allows of greatly simplifying the analysis of such models. Nevertheless it seems that the reason why the problem of the existence of non-basics has not previously arisen is also due to the fact that the concern of those dealing with the input-output analysis has remained for the most part concentrated on the aspect of production levels rather than on the aspect of prices. When we consider the production levels there arises the problem of determining the conditions for a production system to be *technically* viable or, that is to say, to be able to produce a net positive surplus of *all* goods[1]. Such conditions are the same whether the technology matrices are indecomposable or they are not. In other words, when we consider production levels the presence of non-basic commodities does not create particular problems. On the other hand the problem arises when we consider, as Sraffa has done, the aspect of prices. As we shall show in the appendix, the conditions of technical viability are satisfied also by Sraffa's model with surplus. But they are not always able to guarantee that the prices are strictly positive. In fact, as Newman has pointed out, it may happen that some prices are negative. One solution has been provided by Newman himself when he states that the question of non-basics "is partly a matter of the degree of aggregation"[2]. This is, however, unacceptable because the above-mentioned consideration has an exclusively empirical relevance. In principle a multi-sectoral theoretical scheme must be able to represent the economic reality whatever its degree of disaggregation.

Rather than get rid of all non-basic commodities that, as we have already said, represent an important category of goods, it seems preferable to me to follow a course that does not exclude such commodities *a priori*. It is worth noticing that the distinction of goods into basics and non-basics is so much more important for Sraffa because it can be traced back to Ricardo[3] to whom, as Newman explicitly acknowledges, Sraffa's whole book is indebted.

[1] Such conditions have been determined by *Hawkins* and *Simon*, Note: Some Conditions of Macroeconomic Stability, Econometrica, 1949, n. 3 and 4. Non-viable production systems are not theoretically relevant because they cannot constitute the basis of a real economic system. In these systems some commodities absorb in their production so much of themselves that the quantities produced are not even sufficient to compensate for the direct and indirect use of the same commodities necessary to maintain those levels of production.

[2] *Newman*, op. cit., p. 67.

[3] See Luigi *Pasinetti*, A Mathematical Formulation of the Ricardian System, Review of Economic Studies, October 1959, p. 85. The distinction of goods into basics and non-basics is important because the rate of profit depends on the production conditions of
258 the first, but not of the second.

The problem of non-basic commodities can be resolved, in my opinion, only by explaining the economic meaning of the conditions that must be satisfied by non-basic industries in order that prices are all positive. The main purpose of this note is precisely that of determining the above-mentioned conditions and to show, contrary to Newman's opinion, that they have a precise and clear economic meaning, depending on the assumption of a uniform rate of profit.

In the last paragraph we shall consider a characteristic of non-basic commodities which arises when, in the field of input-output analysis, we deal with production levels instead of prices. Such a characteristic is interesting because it is diametrically opposed to what is manifested when we consider prices.

In order not to overload the discussion with too lengthy proofs we shall follow the criterion of referring to the appendix.

2[1]. For non-basic commodities which require for their proper production only basic commodities the problem discussed above does not arise at all. Let us consider the equation relative to one of such non-basics. It is immediately evident that the only variable is constituted by the price of the non-basic commodity in question. The other variables that are involved, that is to say, the prices of basics and the rate of profit, are determined in a completely independent way by the equations relative to the group of basic commodities. Since all prices of basics and the rate of profit are positive, and since at least one basic commodity enters into the production of each commodity, the price of the considered non-basic is *a fortiori* positive. Therefore our question concerns only those non-basics that are used in their own production and/or in the production of an inter-connected group of non-basics. Let us assume then, just for simplicity, that the system is constituted by the group of basic and by only one group of non-basic products. Referring the reader to the appendix for the proof, the necessary and sufficient condition that all prices be strictly positive may be expressed through the following inequality

$$s^* > r^* \tag{1}$$

where r^* is the rate of profit of the system determined by the group of basics and s^* is the rate of profit that would be determined in the group of non-basic industries if the prices of basic commodities were all zero, that is, if the basic products were free.

To simplify the discussion, let us consider a system constituted by two industries: the first non-basic and the second basic

[1] We assume that the reader is familiar with the main characteristics of Sraffa's systems. In this paragraph we are mainly concerned with the system with surplus and subsistence wage-rate (that is Newman's S2).

$$\begin{aligned} (a_{11} + a_{12}\, p_2)\,(1 + r) &= 1 \\ a_{22}\, p_2\,(1 + r) &= p_2 \end{aligned} \tag{2}$$

in which the symbol a_{ij} indicates the quantity of good j ($j = 1, 2$) used in the production of the total output of good i ($i = 1, 2$), taken as unit of measuremet[1]. The first product has been chosen as *numéraire,* i. e. $p_1 \equiv 1$. The rate of profit of the system is determined by the basic industry. We have in fact

$$r^* = \frac{1 - a_{22}}{a_{22}}. \tag{3}$$

Such rate is given by the ratio of the surplus of the basic product $(1 - a_{22})$ to the means of production (a_{22}) both the quantities being expressed in homogeneous physical terms[2]. The own rate of profit of the non-basic product is

$$\frac{1 - a_{11} - a_{12}\, p_2}{a_{11} + a_{12}\, p_2} \tag{4}$$

which not only depends on the quantity a_{11} with which the non-basic commodity enters into its own production but also on the quantity a_{12} with which the basic commodity enters into the production of the non-basic and on the price p_2 of the basic product. If we let p_2 tend to zero, then eventually also the profit of the non-basic industry is expressed in physical terms. We have

$$\lim_{p_2 \to 0} \frac{1 - a_{11} - a_{12}\, p_2}{a_{11} + a_{12}\, p_2} = \frac{1 - a_{11}}{a_{11}} = s^* \tag{5}$$

where s^* is the maximum rate of profit realizable by the non-basic industry compatibly with its technical structure. If $a_{11} > a_{22}$ it immediately follows that

$$s^* = \frac{1 - a_{11}}{a_{11}} < \frac{1 - a_{22}}{a_{22}} = r^*. \tag{6}$$

This means that, if the proportion with which the non-basic commodity enters into its own production is greater than the proportion with which the basic commodity enters into its own, then the maximum rate of profit realizable by the non-basic industry is less than the rate of profit of the basic industry that determines the rate of profit of the system. In other words, the non-basic industry is *economically* non-viable in the sense that it is not able to keep up the

[1] The meaning of the symbols is that given them by *Newman,* op. cit.

[2] This reminds us of Ricardo's "corn and iron system", where the rate of profit is determined by the corn industry. See *Newman,* op. cit., p. 62, and Sraffa's Introduction to the Work and Correspondence of David Ricardo ed. P. *Sraffa,* vol. I, Cambridge
260 University Press 1951, pp. XXX–XXXIII.

degree of profitability imposed by the basic industry. To realize the rate of profit of the system it should be able to get the basic commodity necessary to its production at a negative price (if $a_{11} > a_{22}$) or, at most, zero (if $a_{11} = a_{22}$). Therefore we may conclude that, if $a_{11} \geqq a_{22}$, there is no positive value of price p_2 of the basic commodity which can permit a uniform rate of profit. If, on the other hand, $a_{11} < a_{22}$, such value always exists. In fact, by imposing the equality of the rates of profit of the two industries, which is the fundamental assumption on the basis of which Sraffa constructed his price system, we have

$$\frac{1 - a_{22}}{a_{22}} = \frac{1 - a_{11} - a_{12} p_2}{a_{11} + a_{12} p_2} \tag{7}$$

from which

$$p_2 = \frac{a_{22} - a_{11}}{a_{12}}. \tag{8}$$

Recalling that $a_{12} > 0$ we can state that in the system there obtains a uniform rate of profit and strictly positive prices if, and only if,

$$a_{11} < a_{22}. \tag{9}$$

The (9), which is none other than a particular case of (1), corresponds to the condition enunciated by Newman on p. 67. It has a clear economic meaning since it expresses the restraint that the non-basic industry must satisfy in order to produce at the same conditions of profitability of the basic industry. This situation is a direct consequence of the different nature of basic and non-basic commodities. As non-basics do not enter into the production of basics, they have no influence in the determination of the prices of basic products and of the rate of profit which on the contrary are completely determined by the production conditions of the basics themselves. Because on the other hand the rate of profit is, by hypothesis, uniform in the system, the non-basic industries are compelled to accept the rate of profit which has been independently determined in the group of the basic industries. The fact, however, that they *must* accept it, does not mean that they *can* accept it. They can accept it if, and only if, their structure satisfies condition (1).

We can synthesize what has been said above in the following way. When in the system there exist only basic commodities, the conditions of *technical* viability are necessary and sufficient to guarantee the positivity of the rate of profit and of prices. When there are also non-basic commodities, these conditions are still necessary, but no longer sufficient. We must then find stronger conditions. Such conditions, necessary *and* sufficient, are represented by (1), which I call "condition of *economic* viability of the non-basic industries".

Till now we have been dealing with Sraffa's model with surplus in which wages consist of the necessary subsistence of workers. Newman has considered also Sraffa's model in which wages may rise above the subsistence level and in which, therefore, workers as well as capitalists may get a share of the surplus. He has, however, dealt with this model on the assumption that non-basics do not exist. To complete the discussion on non-basic commodities in the case of simple production it is enough to show that Newman's proofs apply even when non-basics are introduced, provided that they satisfy a condition analogous to that discussed above. Furthermore it is possible to show that, in the sense that will be explained in the appendix, this condition is less stringent than that valid in the case of subsistence wage-rate.

It could, at first sight, seem strange that the question of non-basic commodities has arisen in Sraffa's book in which the phenomenon of consumption, in harmony with the classical tradition, plays no part, and not in the province of neo-classical theory for which, on the contrary, consumption is very important. The fact is, however, symptomatic that Sraffa uses the distinction of products into basics and non-basics essentially to show how consumer goods play no part in the determination of the rate of profit and of the prices of the other products. And this is perfectly in line with classical thought, especially with Ricardo and Marx[1].

3. In this paragraph we shall introduce the concept of basic and non-basic commodities into the mechanism of the input-output analysis to bring into relief a characteristic of these products opposite to what is manifested when we consider prices.

Assuming fixed technical coefficients, the fundamental problem of the input-output analysis is that of determining the gross outputs of the various goods necessary to obtain certain given quantities of net outputs of the same goods. For example, if we consider a system constituted by only two industries, the above-mentioned problem consists in determining the levels x_1 and x_2 of the first and of the second industry necessary to yield the given final net quantities c_1 and c_2 of the goods produced by the two industries:

$$\begin{aligned} x_1 &= a_{11}\,x_1 + a_{21}\,x_2 + c_1 \\ x_2 &= a_{12}\,x_1 + a_{22}\,x_2 + c_2 \end{aligned} \tag{10}$$

where a_{ij} is a usual input coefficient.

If the first industry is non-basic, that is if $a_{21} = 0$, it is immediately evident that the first equation of system (10) is independent of the other equation in the

[1] See *Pasinetti*, op. cit., p. 85, and L. *von Bortkiewicz*, Zur Berichtigung der grundlegenden theoretischen Konstruktion von Marx im dritten Band des «Kapital», Jahrbücher für Nationalökonomie und Statistik, July 1907. See also *Sraffa*, op. cit., pp. 7–8 and p. 54.

sense that it permits to directly determine the level at which the non-basic good must be produced to obtain the final net quantity c_1 of such a good. The level of the second industry, the basic one, depends instead on the level of the non-basic one.

In general, we can say that variations in the final net outputs of non-basics alter the levels of basics, because basics enter into the production of non-basics. On the contrary, variations in the final net outputs of basics do not alter the levels of the production of the non-basics, because non-basics do not enter into the production of basics[1].

These characteristics of basics and non-basics in the system of levels are precisely the *dual* aspect of the characteristics of the same goods in the price system.

Mathematical Appendix

1. Since the quantities of commodities used and produced are given, it will be convenient to follow Newman's example and to take the total output of each commodity as unity. Then Sraffa's system S2 becomes

$$Ap\,(1 + r) = p \tag{11}$$

where a_{ij} = element (i, j) of the non-negative matrix A; it is the proportion of the output of commodity j used in the production of one unit of commodity i $(i, j = 1, 2, ..., n)$.

p_j = price of one unit of commodity j $(j = 1, 2, ..., n)$, which, for the chosen unit of measurement, equals the value of the total output of commodity j. It is the element j of the column vector p.

r = rate of profit.

Because of the presence of non-basic commodities, the matrix A is decomposable. With appropriate changes in the commodities subscripts, it generally takes the form

$$\begin{bmatrix} A_b & 0 & 0 & \dots & 0 \\ A_{b_1} & A_{n_1} & 0 & \dots & 0 \\ A_{b_2} & A_{n_{21}} & A_{n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{b_k} & A_{n_{k1}} & A_{n_{k2}} & \dots & A_{n_k} \end{bmatrix} \tag{12}$$

[1] O. *Lange* (Introduction to Econometrics, Pergamon Press 1959, pp. 242–244) with a different terminology remarks that when a matrix is decomposable, it is possible to "split the system into two or more systems of unilaterally dependent equations". But then he gives some examples in which he seems to believe that basic goods are independent of the levels of non-basics and not vice versa. This misunderstanding stems from 263

where A_b = indecomposable square matrix of coefficients for industries producing basic commodities; this matrix may consist of one single (non-zero) element[1].

A_{n_v} = indecomposable square matrix of coefficients for industries producing the commodities of group v of non-basics ($v = 1, 2, \ldots, k$). These matrices may consist of only one single (possibly zero) element.

A_{b_v} = matrix of coefficients for basic commodities entering into the production of group v of non-basics ($v = 1, 2, \ldots, k$). These matrices are generally rectangular and must contain at least one non-zero element.

$A_{n_{uv}}$ = matrix of coefficients for non-basic commodities of the group v entering into the production of the group u of non-basics ($v = 1, 2, \ldots, k - 1$; $u = v + 1, v + 2, \ldots, k$). These matrices are generally rectangular and most of them consist entirely of zero elements. We may reasonably expect that there will be an integer h ($1 \leqq h \leqq k - 1$), such that $A_{n_{uv}} = 0$ for $v \geqq h$. Actually most of the groups of non-basics are not connected with other groups of non-basics[2].

The zeros indicate rectangular or square matrices of appropriate order consisting entirely of zero elements.

Sraffa assumes that the system is in a self-replacing state, namely $\sum_{i=1}^{n} a_{ij} \leqq 1$ ($j = 1, 2, \ldots, n$), and that there is at least one j such that $\sum_{i=1}^{n} a_{ij} < 1$. Besides, he implicitly assumes that every sub-matrix A_{n_v} with $v \geqq h$ has at least one column the sum of which is strictly less than unity[3]. We will call this set of propositions about the matrix A as assumption H.

the fact that while he was discussing the open input-output model in which the data are the final net outputs, Lange had in mind the opposite method of considering as given the gross outputs of some basic industries (electricity etc.) and as variables the net outputs of some non-basic goods (consumer goods).

[1] In other words, the system contains at least one basic commodity.

[2] If m_b is the number of basics and m_v the number of non-basics of group v, of course the relation $m_b + m_1 + \ldots + m_k = n$ must hold.

[3] The fact that all the columns of a sub-matrix A_{n_v} ($v \geqq h$) have a sum equal to unity implies that all commodities of the group v of non-basics absorb so much of themselves that no surplus is left. The existence of such a group of non-basics, the production of which necessarily requires also basic commodities, is economic nonsense. Sraffa implicitly excludes this case, when he says (op.cit., p.7) that "one effect of the emergence of a surplus" is that "there is room for a new class of 'luxury' products" (non-basics). In other words the production of non-basics has sense only if they are capable of giving a net surplus.

It is possible to see immediately that the conditions of technical viability are satisfied by Sraffa's system. These conditions require that all the characteristic roots of A be less than one in modulus. Because of assumption H, the corollary to Solow's first theorem[1] assures that the matrix (12) satisfies such conditions.

If there are only basic commodities, that is the matrix A is indecomposable, the conditions of technical viability are sufficient in order to assure the positivity of all prices and of the rate of profit. System (11) may be written

$$Ap = \frac{1}{1+r}\, p\,. \tag{13}$$

Applying Frobenius theorem[2], we know that there exists a positive $\frac{1}{1+r^*}$ to which corresponds a positive p^*. Because of the conditions of technical viability we must have $\frac{1}{1+r^*} < 1$, from which it immediately follows that $r^* > 0$.

2. To determine the necessary and sufficient condition that a system containing non-basic commodities has all positive prices, we assume for simplicity's sake that there is only one group of non-basics[3]. We partition A and p into

$$A = \begin{array}{c} \left[\begin{array}{cc} A_b & 0 \\ A_{b_1} & A_{n_1} \end{array}\right] \begin{array}{l} \}\, m_b \\ \}\, n-m_b \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{m_b} & \underbrace{\qquad}_{n-m_b} \end{array} \end{array} \qquad p = \left[\begin{array}{c} p_b \\ p_{n_1} \end{array}\right] \begin{array}{l} \}\, m_b \\ \}\, n-m_b \end{array} \tag{14}$$

Then the system (11) becomes

$$A_b\, p_b(1+r) = p_b \tag{15}$$

$$A_{b_1}\, p_b(1+r) + A_{n_1}\, p_{n_1}(1+r) = p_{n_1}\,. \tag{16}$$

System (11) has been divided into two parts (15) and (16), of which the first is completely independent of the second. Taking $p_1 \equiv 1$, (15) may be solved with respect to the m_b variables $p_2, \ldots, p_{m_b}$ and r. The equilibrium values p_b^* and r^* are all positive because of the indecomposability of A_b. Substituting p_b^* and r^* in (16) and putting

$$A_{b_1}\, p_b^* = c \tag{17}$$

we obtain from (16)

[1] Robert *Solow*, On the Structure of Linear Models, Econometrica, 1952, n. 1.

[2] Theorem I of G. *Debreu* and I. *Herstein*, Nonnegative Square Matrices, Econometrica, October 1953, p. 598.

[3] The proof can be immediately generalised.

$$p^*_{n_1} = \left(\mathrm{I} \frac{1}{1+r^*} - A_{n_1} \right)^{-1} c. \tag{18}$$

Since $c \geqq 0$ and $c \neq 0$, $p^*_{n_1}$ is positive if, and only if, $\left(\mathrm{I} \frac{1}{1+r^*} - A_{n_1} \right)^{-1}$ is positive. From theorem III of Debreu and Herstein[1], it follows that $\left(\mathrm{I} \frac{1}{1+r^*} - A_{n_1} \right)^{-1} > 0$ if, and only if $\frac{1}{1+r^*} > \frac{1}{1+s^*}$, where $\frac{1}{1+s^*}$ is the dominant eigenvalue of the indecomposable non-negative matrix A_{n_1}, namely if, and only if, $s^* > r^*$[2].

3. Let us consider now system S3 in which wages are above the subsistence level and are represented explicitely by the variable w, indicating the wage per unit of labour, that is the wage rate. S3 is

$$Ap(1+r) + wL = p \tag{19}$$

where L is the column vector of the inputs of labour $l_i (i = 1, 2, \ldots, n)$. We can partition A and p, as in (14). System (19) then becomes

$$A_b \, p_b(1+r) + wL_b = p_b \tag{20}$$

$$A_{b_1} \, p_b(1+r) + A_{n_1} \, p_{n_1}(1+r) + wL_{n_1} = p_{n_1} \tag{21}$$

where L_b is the vector of direct labour inputs of basics and L_{n_1} that of non-basics. In S3 we can consider r as datum. As Newman has proved, in order that w and p_b be positive, r must be less than r^*[3]. Supposing that actually $r < r^*$, (20) has the positive solutions w^{**} and p_b^{**}. Proceeding as in the preceding paragraph, we will find that $p^{**}_{n_1}$ is positive if $s^* > r$[4]. This condition is weaker than that in paragraph 2 in the sense that there is always an interval $(0, s^*)$ of the rate of profit r such that, if r is inside it, $p^{**}_{n_1}$ is positive. But $p^{**}_{n_1} > 0$ for the whole interval $(0, r^*)$ of r if, and only if, $s^* > r^*$.

1 *Debreu* and *Herstein*, op. cit., p. 602.

2 When I proved this theorem, I did not know the existence of Gantmacher's proof (in Theory of Matrices, New York, 1959). Since the condition of economic viability of non-basics plays a crucial part in this paper, I have decided to leave this proof for completeness. Besides, it helps to abbreviate the proof in the following paragraph.

3 $1/1 + r^*$ is the dominant eigenvalue of A_b.

4 $1/1 + s^*$ is the dominant eigenvalue of A_{n_1}.

Part III
On the History of Economic Thought

[22]

ON THE SIGNIFICANCE OF RECENT CONTROVERSIES ON CAPITAL THEORY: A MARXIAN VIEW[1]

I

RECENT controversies on capital theory between the Cambridge School and the so-called Neo-classical School centre on the question of treating "capital" as a "factor of production" for a theory of distribution in a capitalist economy. It must be emphasised that questions like the measurement of "capital" are, as such, not central to the controversy, but assume relevance in so far as they have a direct bearing on the theory of distribution. Since the rate of profit is a pure number per unit of time, distribution of income between "profits" and "wages" must reckon "capital" in the same unit in which wages and income are measured. Consequently, a valuation problem arises unless by assumption "capital" consists of the stock of the same commodity in which wages are paid, *i.e.*, a one-commodity world. And once this valuation problem is faced, the foundations of a Neo-classical parable in which the magnitude of "capital" as a "factor of production" is independent of the distribution of income become logically insecure. This is a classical problem in economic theory: Ricardo recognised that, even in his circulating-capital model, the pattern of relative prices corresponds to the ruling (uniform) rate of profit, and consequently, the valuation of commodities entering the production process as "means of production" is not independent of that ruling rate of profit. Wicksell attacked the Austrian attempt to measure "capital-intensity" in terms of the "average period of production." He realised that no such "time-measure" of "capital" was possible independent of the rate of profit. More recently, Piero Sraffa and Joan Robinson have produced logical arguments emphasising the limitations of a Neo-classical parable. Thus, the rate of profit may be *positively* correlated to the value of capital per man (*i.e.*, the negative Wicksell-Effect) running counter to the Neo-classical story, or what is still more disastrous for the Neo-classical parable, the *same* technique of production may be competitive both at a relatively "high" and a "low" rate of profit, but dominated by a different technique for the interim values of the rate of profit (*i.e.*, the reswitching of techniques).

Still, then, one may feel, and indeed it is often argued, that all this trouble arising from the valuation of "capital" is no more than a usual index-number problem. Since ideal index-numbers are hard to find, one should let the matter rest at that and accept the simple-minded Neo-

[1] I am indebted to Joan Robinson, Krishna Bharadwaj, Khaleeq Naqvi and Donald Harris for helpful comments and long discussions on the subject.

classical parable as the *approximate* basis for a theory of distribution in a profit–wage economy. The present paper argues that under the obvious surface of an index-number problem deeper issues lie in connection with the valuation of " capital." It is better to face these issues and re-examine the teachings of conventional theory than to dodge them as mere index-number problems. In the view of the present writer, Marx's understanding of the role of " capital " in the capitalist mode of production focuses attention on some of these central issues, which to the less sophisticated appear no more than yet another index-number problem.

II

In his famous Introduction to the Critique of Political Economy, Marx drew an interesting analogy between *language* as a system of communication and the *social* organisation of production. Like language, Marx claimed, economic production must be viewed in the context of a social organisation: " Production by isolated individuals outside a society . . . is an absurdity as the idea of development of language without individuals living together and talking to one another." [1] Yet there are certain features common to languages of varying degrees of complexity—from the most " primitive " to the more " subtle "—which makes language as a *general* concept useful. The same is true of various types of economic organisation geared to production. They also exhibit common features which lend themselves to abstraction in terms of general concepts. But the failure to recognise the major points of departure which differentiate one economic organisation from another, is according to Marx, the basic source of confusion in Political Economy.[2] Marx's own distinction between the " forces of production " and the " relations of production " is relevant here. The former concept relates to man's relation to nature and technology while the latter corresponds to man's relation to man in a social organisation of production. Each type of economic organisation develops its own " relations of production " or " rules of the game," often sanctioned by law or religion. Marx insisted that concepts that are useful in Political Economy must take into account these " rules of the game." This methodological position gives rise to the notion of Marxian " *categories* " by which abstract or general concepts are restricted to a specific set of " rules of the game." Economic theory, which ignores such " rules of the game " and works in terms of general features only, runs the danger of being totally a-historical in spirit.

Throughout his work, Marx maintained this methodological position in defining the notion of " capital." Taken out of a particular form of econo-

[1] *A Contribution to the Critique of Political Economy* (translated from the second German edition by N. I. Stoke), p. 268. Referred to as " Critique " later.

[2] Thus Marx satirically writes, " The failure to remember this one fact is the source of all wisdom of modern economists . . .", *Critique, op. cit.*, p. 269.

mic organisation, the notion of "capital" reduces to the idea of mere physical instruments of production or "stored-up impersonal labour." To use this notion of "capital" holding in the abstract in the context of a particular economic organisation, *e.g.*, the capitalistic mode of production, can be thoroughly misleading if it does not reflect the "relations of production" which characterise a capitalist economy. Consequently, Marx emphasised that "capital" in the context of the capitalistic rules of the game is also a *social relation* for commanding labour and generating surplus value. He categorically states, "The means of production become capital only in so far as they have become separated from the labourer and confront labour as an independent power."[1] In other words, means of production is *not* "capital" unless owned by non-labourers. This emphasises the relevant aspect of "capital" for a theory of distribution: as a means for generating surplus value by exploiting live labour, capital is also a source of surplus value and income to the capitalists. Thus, "capital" as a Marxian "category" notion is: (*a*) an instrument of production—a pure physical object (belonging to the Marxian notion of "forces of production"); and (*b*) a social ownership relation giving rise to capitalists' income (belonging to the Marxian notion of "relations of production"). Taken out of the specific context of capitalistic mode of production, the last feature may disappear.

It must be granted that Marx himself was unable to indicate the *logical* implications of his understanding of the role of "capital" for the formulation of a theory of distribution between profits and wages in a capitalistic economy. In the view of the present writer this is precisely what the recent controversies on capital theory do: they lay bare the *logical* weaknesses of treating "capital" merely as an instrument of production in developing a theory of distribution in a capitalist economy.

The central consequence of treating "capital" as a mere physical instrument of production results in the prevalent Neo-classical methodology of treating "production" and "distribution" as two separable branches of inquiry. The conventional "production function" is supposed to depict the pure production aspect of an economy and the profit-maximising behaviour leading to marginal calculations gives a corresponding "marginal productivity theory" of distribution. The single most important consequence of accepting the Marxian definition of "capital," on the other hand, is to

[1] *Theories of Surplus Value*, Part I (Moscow: Foreign Publishing House), p. 396. Compare also, ". . . this brings to completion the fetishism peculiar to bourgeois Political Economy, the fetishism which metamorphoses the social, economic character impressed on things in the process of social production, into a natural character stemming from the material nature of those things. For instance, "instruments of labour are fixed capital" is a scholastic definition, which leads to contradictions and confusion . . . instruments of labour are fixed capital only if the process of production is really a capitalist process of production and the means of production are therefore really capital and possess economic definiteness, the social character of capital. . . . If not, they remain instruments of labour, without being fixed capital." K. Marx, *Capital*, Volume II (Moscow: Foreign Publishing House, pp. 225–6.)

recognise the logical untenability of the separation between "production" and "distribution" in a general conceptual scheme. To this central theme of capital theory we turn in the next section (Section III) and show its connection with the Marxian position on "capital" in the last section (Section IV).

III

The force of the argument that the separation between "production" and "distribution" is an artificial one can be analytically demonstrated by starting with a definitional relation of the distribution of national income.[1]

Let $Y =$ net national income measured in a homogeneous consumption good;

$K =$ value of capital in terms of the same consumption good; [2]

$L =$ number of employed workers;

$r =$ the rate of profit, a pure number per unit of time; and

$w =$ real wage-rate per worker per unit of time in terms of the consumption good.

Assuming that the net national income is distributed between profits and wages, we have a definitional relation,

$$Y = Kr + Lw \quad \ldots\ldots \quad \text{(i)}$$

Without any loss of generality we can normalise relation (i) by setting $L = 1$ and write in per worker measure

$$y = kr + w \quad \ldots\ldots \quad \text{(ii)}$$

Since relation (ii) is purely definitional in character, it should hold for all economies where net income is being distributed between profits and wages. Consequently, it should be compatible with any acceptable treatment of "capital" in a theory of distribution including the "marginal productivity theory." Unfortunately this is not true in general. In order to see this, we may notionally compare two hypothetical economies—"marginally" different in terms of output per head (y), value of capital per head (k) and their respective profit-wage configurations (r and w). This is obtained by totally differentiating relation (ii), which gives

$$dy = r \,.\, dk + k \,.\, dr + dw \quad \ldots\ldots \quad \text{(iii)}$$

It is clear that the "marginal product of capital," *i.e.*, $\frac{dy}{dk}$, as derived from the definitional equation (iii) above, *does not in general equal* the rate of profit.

[1] The following algebraic argument depends heavily on my article, "The Concept of the Marginal Productivity of Capital and the Wicksell Effect," *Oxford Economic Papers*, 1966.

[2] This is value net of depreciation.

Once this general point is realised, various *special* constructions can be erected which would restore the " marginal productivity " relation by showing that the treatment of " capital " as a " factor of production " is compatible with the definitional distribution relation (iii) above. Two such noted attempts of recent years are due to Mr. D. G. Champernowne [1] and Professor P. A. Samuelson.[2] It is worth our while to examine the essence of their arguments briefly in order to realise their significance in relation to this central question of capital theory (*i.e.*, the separation between " production " and " distribution ").

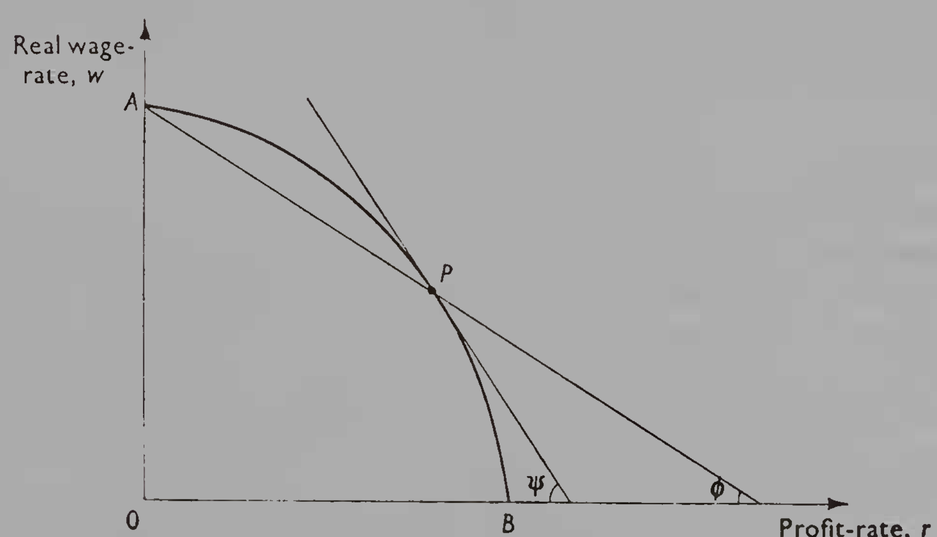
Looking back at relation (iii), it is clear that the " marginal productivity " relation will hold provided, by fluke or *by assumption*,

$$(k \,.\, dr + dw) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

which in turn implies

$$-\frac{dw}{dr} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

Equation (v) can be seen to be equivalent to Professor Samuelson's condition that the elasticity of the " factor-price frontier " equals the distributive shares, when the factors are paid according to their marginal products in an economy with a homogeneous production function of degree one in labour and " capital." Professor Piero Garignani of Italy in an as yet unpublished paper has indicated how special this condition is.[3] Since it fits in well with our algebraic formulation, I reproduce his diagram below:



[1] " The Production Function and the Theory of Capital: A Comment," *Review of Economic Studies*, 1953–54.

[2] " Parable and Realism in Capital Theory: The Surrogate Production Function," *Review of Economic Studies*, 1962.

[3] The argument is based on a seminar given by Professor Garignani in Cambridge in the summer of 1967.

In the figure the wage–profit frontier for a technique is represented by the curve AB, and the output per worker associated with that technique is represented by OA. At a profit–wage configuration given by point P on this frontier the value of capital per worker according to Samuelson's condition (*i.e.*, condition (v)) corresponds to the slope of the tangent at P, *i.e.*, $\tan\psi$. The value of capital following from the definitional relation (ii), on the other hand (*i.e.*, $k = \frac{y-w}{r}$, $r > 0$ and, $y = w_{\max} = OA$ at $r = 0$), is given by $\tan\theta$, which is the slope of the line joining the points A and P. Thus, for *non-linear* wage–profit frontiers $\tan\psi$ and $\tan\theta$ are not equal in general, and Professor Samuelson needs a *linear* relation between profit and wage to make these two values of capital per worker (*i.e.*, $\tan\psi$ and $\tan\theta$) equal. Unfortunately he achieves this by merely fixing his assumptions suitably, and this makes the " Surrogate Capital " case a *special* model of very limited interest. The assumption which Professor Samuelson makes to produce a straight-line frontier is the *uniform* " capital–labour ratio " in all lines of production. In Marx's terminology this is equivalent to the assumption of *uniform* " organic composition of capital " in all lines of production [1]—exactly the assumption which Marx himself made in the first two volumes of his " Capital " to avoid the famous " Transformation Problem " that appears only in the third volume. Professor Samuelson rediscovered the importance of this assumption about a hundred years later!

A more imaginative attempt to restore the " marginal productivity relation " is to be found in Mr. Champernowne's " Chain-Index of Capital." If we differentiate (ii) for *given parametric values* of $r = \bar{r}$ and $w = \bar{w}$, then instead of the previous relation (iii), we obtain

$$dy = \bar{r}\,.\,dk, \text{ i.e., } \frac{dy}{dk} = \bar{r} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(vi)}$$

Economically, equation (vi) compares two situations, both characterised by the *same given* profit–wage configuration $(\bar{r}, \bar{w})$. Consequently, equation (vi) will continue to hold in the situation where two alternative methods of production are equally profitable at a *given* rate of profit so that the differences in the value of output per worker is exactly compensated for by the difference in the value of capital per worker to keep the *given* profit–wage configuration undisturbed.[2] This enables Mr. Champernowne to arrange all the alternative methods of production in a " chain " or " sequence " (under some restrictive assumptions) such that any two consecutive techniques in that sequence are equally profitable for a *given* rate of profit.

[1] The " direct " and " indirect " labour component is valued at a *zero* rate of profit in accordance with the " labour theory of value."

[2] In recent jargon this is often described as a " switch-point," when two techniques are equally profitable at some given profit-rate.

Consequently, for each pair of consecutive techniques relation (vi) will satisfy the relevant marginal relation. From this the conventional " production function " can be traced out by a parametric variation of the rate of profit. While Mr. Champernowne was candid enough to admit that his construction fails in case of the possibility of " reswitching of techniques " or when more than two " factors of production " are involved, the central point in this construction lies in the way the rate of profit is treated. The " chain " or the sequential ordering of techniques corresponds to the parametric variation of the *given* rate of profit. In other words, the rate of profit continues to be an *independent* variable of the system (or the corresponding real wage-rate), as is amply demonstrated by relation (vi). The " marginal productivity relation " in (vi) does *not* give a theory of " determination " of the rate of profit in any way, and this brings us back to the main current of the argument regarding the significance of treating " capital " also as a social ownership relation.

IV

For Marx this problem could be posed in a slightly different form. " Capital " as a " factor of production " is a total abstraction without any historical counterpart; it is not a Marxian " category " belonging to a particular historical form of economic organisation. In order to use this abstraction for a theory of distribution relevant to capitalist economies, one must also consider the social ownership aspect of " capital " which allows for the exploitation of live labour and creation of surplus value for capitalists' income, corresponding to a *given* rate of exploitation. For an academic economist, Marx left open the question of how the rate of exploitation is determined. He viewed it himself in terms of the balance of class-forces and, significantly enough, did not try to provide a " theory " of distribution. On a logical plane, however, once the rate of exploitation is given the entire system of relative prices is determined under conditions of competitive equilibrium, and the valuation of capital presents no problem in terms of any chosen numeraire. Whether to take the rate of exploitation as given from outside is essentially a matter of judgment for an academic economist. Alternatively, he could take the rate of profit or the real wage-rate as given.[1]

[1] A large number of Cambridge growth models " close " the system through a relation between the rate of profit and the rate of growth. See Joan Robinson, *The Accumulation of Capital*, Book II (London, 1956), Nicholas Kaldor, "A Model of Economic Growth,' ECONOMIC JOURNAL, 1957, and Luigi Pasinetti, " The Rate of Profit and Income Distribution in relation to the Rate of Economic Growth," *Review of Economic Studies*, 1962.

In Classical Political Economy the system was " closed " through the " iron-law of wages." In Marx's writings there is also the notion of a long-run inflexible real wage-rate maintained through the " reserve army " of labour continuously created through breakdown of " pre-capitalistic modes of production " in the early stages of capitalism and through " labour-saving innovations " at a later stage. It would appear that a *given* real wage-rate together with a *given* rate of exploitation overdetermines the system of equations for relative prices.

But it must be recognised that the system of relative prices (which determines the value of the " means of production " and, consequently, distribution) has a degree of freedom and becomes locked only when either the rate of exploitation or the rate of profit or the real wage-rate is taken as an independent datum. The rest follows as a matter of logic under the competitive assumption of equal rate of profit in all lines of production. The theory of distribution therefore continues to be a matter of *political* economy, simply because one has to form one's judgement regarding how this degree of freedom is closed through the functioning of capitalism. Marx's " relations of production " reflecting the social-ownership aspect of " capital " is unavoidable precisely here. But to pretend that we can still have a theory of distribution independent of such considerations is either a very *special* construction or faulty logic.

Amit Bhaduri

Delhi School of Economics.

[23]

MR. SRAFFA'S REHABILITATION OF CLASSICAL ECONOMICS[1]

I

MR. Sraffa's important new book, *Production of Commodities by Means of Commodities,*[2] is described in the publisher's blurb as 'a work of a specialist character, addressed to those interested in pure economic theory'. One should not be intimidated by this, however: the book is a short one, running to less than 100 pages; the argument is on the whole quite lucid; and the mathematics used is of a very elementary character. Nevertheless, the problems with which the book deals are from their very nature rather complex and abstract; and some of Sraffa's analytical tools and methods are likely to appear strange to those unversed in the ways of Ricardo and Marx. The present article is an attempt to summarise Sraffa's main argument and to state in simple language just what I think he is getting at.

The book can be looked at from various points of view. It can be regarded, if one pleases, simply as an unorthodox theoretical model of a particular type of economy, designed to solve the traditional problem of value in a new way—in which case it will be judged purely on its own merits. Or it can be regarded as an implicit attack on modern marginal analysis: the sub-title of the book is 'Prelude to a Critique of Economic Theory', and Sraffa in his preface expresses the hope that someone will eventually attempt the job of basing a critique of the marginal analysis on his foundations. Or, finally, the book can be regarded as a sort of magnificent rehabilitation of the classical approach to certain crucial problems relating to value and distribution. It is upon this third aspect of the book that I am concentrating in the present article. In doing so, I do not of course want to suggest that the *essence* of Sraffa's book lies in this rehabilitation of the classical approach: Sraffa's primary aim is not to show that there's life in the old dogs yet, but to build a 20th-century model to deal with 20th-century problems. I am approaching his book in

[1] This article is based on a lecture given at the University College of North Wales, Bangor, on 21 November 1960. It owes much to the criticisms of Mr. Maurice Dobb and Mr. John Eaton, who must not, however, be held responsible for any errors which remain.

[2] Cambridge University Press, 1960. 12s. 6d.

this particular way simply because I think it affords the best method of understanding Sraffa's basic argument.

Let me begin by making three general points about the relation between Sraffa's model and the old classical models. First, both Sraffa's model and the classical models are concerned with the investigation of one and the same set of properties of an economic system—those properties, as Sraffa puts it, which 'do not depend on changes in the scale of production or in the proportions of "factors"'.[3] The classical people, at any rate in their basic analysis of the economy as such, were usually *in effect* concerned with these properties alone, since they often tended to assume that under given technological conditions returns to scale for the industry as a whole would be constant,[4] and that the proportions in which the different means of production were used in an industry would be technically fixed. Sraffa, by way of distinction, makes no assumption whatever about the variability or constancy of returns. Rather, he simply selects for analysis a particular kind of economic system in which the question of whether returns are variable or constant is irrelevant. This system is one in which production goes on from day to day and from year to year in exactly the same way, without any changes in scale or 'factor' proportions at all. By this means Sraffa is able *deliberately* to concern himself with the investigation of the same properties of an economic system which the classical people *objectively* concerned themselves with, while at the same time avoiding the necessity of making any (possibly objectionable) assumptions about the nature of returns.

The second point is this: The classical people, anxious as they were to propound generalised statements or 'laws' relating to the economy in which they were interested, naturally wanted to make their systems 'determinate' in some useful and meaningful sense of that word. The methods they employed to secure the requisite degree of determinacy were often ingenious and stimulating. But they did not hit on the idea that it would help greatly to secure determinacy if certain specific interrelations were postulated between elements of input and elements of output over the economy as a whole, so that the output of certain industries was assumed to constitute the input of others. They were of course aware that such interrelations did exist and were important: Quesnay, after all, framed his *Tableau Economique;* Marx worked out his famous reproduction schemes; and

[3] Sraffa, p. v.

[4] I.e., roughly, that a change in the size of an industry would not alter the unit cost of producing the commodity concerned.

Ricardo (if Sraffa is right) held at one stage a 'corn-ratio theory of profits'.[5] The point I am making is simply that they did not, by and large, use these postulated interrelationships as an integral part of the methods which they employed to make prices and 'factor' incomes determinate—i.e., to solve the general problem of value. And this is precisely what Sraffa *does* do.[6]

The third point is this: The classical people were primarily interested in the problem of the *development* of the capitalist system, but they believed that a necessary preliminary to the study of this problem was an analysis of *the nature of the capitalist system as such*. And the best method of going about this analysis, they believed, was to begin by imagining capitalism suddenly impinging upon a pre-capitalist form of economy in which, in effect, labour was the only 'factor' receiving a reward. In this pre-capitalist economy, which Smith called the 'early and rude state of society' and Marx called 'simple commodity production', the whole produce of labour went to the labourers.[7] In such an economy, it was claimed, the relative equilibrium prices of commodities would tend to be equal to the relative quantities of labour required from first to last to produce them. What happened, then, when a class of capitalists arrived on the scene, and the net product of the economy consequently came to be shared between labourers and capitalists? In particular, what happened to relative equilibrium prices? Did they remain equal to relative quantities of embodied labour, or did they now diverge from these quantities? If they diverged, were the divergences haphazard, or could they be shown to be in some useful sense 'subject to law'? Did the divergences render it necessary to throw out the simple 'law of value' which used to operate in the pre-capitalist economy, or could they be regarded as merely *modifying* its operation? These questions were not regarded as purely academic ones, with little relevance to problems of practical policy. On the contrary, the classical people believed that if one could give adequate answers to them one would then have penetrated to the very essence of the capitalist system, and would be properly equipped to proceed to the major task—that of the determination of what Marx (and Mill) called the 'laws of motion' of the capitalist system. The general procedure which Sraffa adopts, and the questions he asks, are very similar to those I have just described.

[5] Sraffa, p. 93.

[6] The specialist will notice the intellectual affinity between Sraffa's approach and that of the Walrasian-type analysis and modern 'input-output' techniques.

[7] Cf. Adam Smith, *Wealth of Nations* (Cannan edn.), Vol. I, pp. 49 and 66.

II

Sraffa begins with a very simple model of a subsistence economy in which there are only two commodities produced—wheat and iron—and in which the total amount of each of these commodities which goes into the productive process each year is precisely the same as the total amount which comes out.[8] A possible set of conditions of production in such an economy is as follows:

$$\begin{array}{l}
280 \text{ qr. wheat} + 12 \text{ t. iron} \rightarrow 400 \text{ qr. wheat} \\
120 \text{ qr. wheat} + 8 \text{ t. iron} \rightarrow 20 \text{ t. iron} \\
\overline{400} \qquad\qquad\quad \overline{20}
\end{array}$$

In the wheat industry (represented by the first line) 280 quarters of wheat and 12 tons of iron are used up during the year in order to produce an annual output of 400 quarters of wheat. In the iron industry (represented by the second line) 120 quarters of wheat and 8 tons of iron are used up during the year in order to produce an annual output of 20 tons of iron.[9] It will be seen that a total of 400 quarters of wheat and 20 tons of iron goes into the productive process and is used up there, and that 400 quarters of wheat and 20 tons of iron come out of the productive process at the end of the year.

Now, at the end of each year the wheat producers are going to have 400 quarters of wheat in their hands, 280 quarters of which have to be earmarked for the following year's input. If the process of production is to continue from year to year at the same level, it is clear that the proceeds from the sale of the remaining 120 quarters of wheat must be sufficient to enable the wheat producers to buy the 12 tons of iron which they will need as input in the following year. Similarly, the iron producers are going to have 20 tons of iron in their hands, 8 tons of which have to be earmarked for the following year's input. If the process of production is to continue from year to year at the same level, it is clear that the proceeds from the sale of the remaining 12 tons of iron must be sufficient to enable the iron producers to buy the 120 quarters of wheat which they will need as input in the following year. It is evident, therefore, that prices in this economy must be such that 12 tons of iron are exchangeable on the market for 120 quarters of wheat—i.e., that the price of a ton of iron must be ten times the price of a quarter of wheat. This analysis can readily be generalised to cover the case of a more complex subsistence economy

[8] Sraffa, pp. 3-5.

[9] We assume for the moment that subsistence goods for the labourers are included in the wheat and iron inputs.

in which any number k of different commodities is produced. A set of k production equations in price terms can be drawn up in which the number of independent equations is equal to the number of unknowns, so that the prices of all the commodities produced become determinate.[10]

Let us now drop the assumption of a subsistence economy, and turn, as Sraffa does,[11] to the case of an economy in which a surplus over subsistence is yielded. Such an economy might be one with the following conditions of production:

$$
\begin{array}{lll}
280 \text{ qr. wheat} + 12 \text{ t. iron} & \rightarrow & 575 \text{ qr. wheat} \\
120 \text{ qr. wheat} + 8 \text{ t. iron} & \rightarrow & 20 \text{ t. iron} \\
\overline{400} \qquad\qquad\quad \overline{20} & &
\end{array}
$$

It will be seen that this economy is the same as the previous one except that the wheat industry is now assumed to produce 575 quarters of wheat every year instead of 400. If we assume that the rewards going to labour are fully taken care of in the wheat and iron inputs of the two industries, as we have so far been doing,[12] this means that the whole value of the surplus of 175 quarters of wheat will be available for distribution in the form of profit. Let us assume that this profit is distributed in such a way as to make the *rate* of profits equal in both industries—in other words, that the owners of each industry earn what the classical economists called the 'normal' or 'average' rate of profits on their advances. The situation now is that prices must be such as to allow the elements of input in each industry to be replaced, *and* to allow profits on the value of these elements of input to be earned at the same rate in each industry. In the present example, these two conditions will be fulfilled if prices are such as to make 1 ton of iron exchangeable on the market for 15 quarters of wheat, which will bring the average rate of profits out at 25 per cent.[13] Once again this analysis can readily be generalised to cover the case of a more complex economy in which any number k of different commodities is

[10] Specialists will appreciate that any one of the k equations can be inferred from the sum of the others, so that there are in fact only $k-1$ *independent* equations. But it is easy to reduce the number of unknowns to $k-1$ by taking one commodity as the standard of value and making its price equal to unity See Sraffa, p. 5.

[11] Sraffa, pp. 6 ff.

[12] See n. 9.

[13] Let the price of a quarter of wheat be 1; let the price of a ton of iron be p_i; and let the average rate of profits be r. The production equations in price terms will then read as follows:

$$(280+12.p_i)(1+r)=575$$
$$(120+8.p_i)(1+r)=20.p_i$$

These equations yield the solutions $p_i=15$ and $r=\frac{1}{4}$.

produced. A set of *k* production equations in price terms can be drawn up in which the number of independent equations is equal to the number of unknowns, so that the prices of the *k* commodities, *and* the average rate of profits, are all determined.[14]

We must now alter the assumption we have so far been making about wages. Up to this point we have in effect assumed that wages consist of necessary means of subsistence for the workers, and thus, as Sraffa puts it, enter the system 'on the same footing as the fuel for the engines or the feed for the cattle'.[15] But wages may in fact include not only the 'ever-present element of subsistence' (which is constant), but also a 'share of the surplus product' (which is variable).[16] What is one to do about this? The most appropriate thing to do would be to separate the wage into its two component parts, continuing to treat the goods required for the subsistence of the workers as means of production along with the fuel, fodder, etc., and treating the variable element in the wage as a part of the surplus product of the system. Sraffa, however, in order to avoid 'tampering with the traditional wage concept', from now on treats the whole of the wage as variable—i.e., as part of the surplus product. This means that the quantity of labour employed in each industry has from now on to be represented explicitly in our statements of the conditions of production, taking the place of the corresponding quantities of subsistence goods in our previous statements.

When the wage is recognised as containing a variable element, or when, as with Sraffa, the whole of the wage is assumed to be variable, we have another unknown to be added to our list. In a system where *k* commodities are produced, we now have $k+2$ unknowns—the *k* prices, the rate of profits *r*, and the wage *w*.[17] And the best we can do, when we put the production equations in price terms, is to provide $k+1$ equations in order to find these $k+2$ unknowns. Thus the system is not determinate, unless one of the variables can be taken as fixed.[18]

[14] There are *k* independent equations, which, if one commodity is taken as the standard of value and its price made equal to unity, are sufficient to determine the $k-1$ prices and the rate of profits *r*.

[15] Sraffa, p. 9.

[16] This implies that Sraffa is defining the 'surplus product' of a system as the difference between gross output and what Ricardo called 'the absolutely necessary expenses of production.'

[17] 'We suppose labour to be uniform in quality or, what amounts to the same thing, we assume any differences in quality to have been previously reduced to equivalent differences in quantity so that each unit of labour receives the same wage' (Sraffa, p. 10).

[18] Let A be the quantity annually produced of commodity 'a'; let B be the quantity annually produced of commodity 'b'; and so on. Let A_a, B_a, . . . , K_a be the quantities of commodities 'a', 'b', ... 'k' annually used as means of production by the industry which produces A; let A_b, B_b, . . . ,

As Sraffa puts it, 'the system can move with one degree of freedom; and if one of the variables is fixed the others will be fixed too'.[19] In particular, if we know what the wage is, the rate of profits and all the prices will be determined.

III

Having erected this simple model, Sraffa now proceeds to investigate what we have seen to be the basic classical problem—what happens to prices and profits when a class of capitalists arrives on the scene to share the net product of the economy with labour? Sraffa deals with this problem by giving the wage successive values ranging from 1 to 0 —a wage of 1 representing a situation in which *all* the net product goes to labour (i.e., there is no class of capitalists and therefore no profit), and a wage of 0 representing the other extreme in which *none* of the net product goes to labour (i.e., there is a class of capitalists which manages to secure *all* the net product in the form of profit). The main task is to show what happens to prices and to the rate of profits as the wage is reduced from 1 to 0.

When the wage is 1—i.e., when the whole of the net product goes to wages—we in effect revert to the simple set of equations with which we started, except that (as already stated) the quantities of labour employed in each industry will now be shown explicitly in our equations instead of being represented, as they were before, by quantities of subsistence goods. It is easy to show that in this situation the equilibrium prices of the different products will be proportionate to the different quantities of labour which have been directly and indirectly employed to produce them. For if the only form of income is wages,

K_b be the quantities of commodities 'a', 'b', . . . , 'k' annually used as means of production by the industry which produces B; and so on. Let L_a, L_b, . . . , L_k be the quantities of labour annually employed in the industries producing A, B, . . . , K. These quantities are the 'knowns' in our equational system.

Let p_a, p_b, . . . , p_k be the unit prices of commodities 'a', 'b', . . . , 'k'. Let r be the average rate of profits. Let w be the wage per man. These quantities are the 'unknowns' in our equational system.

In generalised form, the production equations in price terms will read as follows:

$$(A_a.p_a + B_a.p_b + \dots + K_a.p_k)(1+r) + L_a.w = A.p_a$$
$$(A_b.p_a + B_b.p_b + \dots + K_b.p_k)(1+r) + L_b.w = B.p_b$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(A_k.p_a + B_k.p_b + \dots + K_k.p_k)(1+r) + L_k.w = K.p_k$$

By putting the surplus product of the system equal to unity (thereby making it the standard in terms of which the wage and the k prices are expressed) we obtain an additional equation, making a total of $k+1$ equations. But there are $k+2$ unknowns, so that the system is not determinate unless one of the variables (e.g. the wage w) can be taken as fixed. (See Sraffa, pp. 4, 6 and 10-11.)

[19] Sraffa, p. 11.

all input-costs ultimately reduce to wage-costs. This means that the value of each end-product will be equal to the sum of its inputs at wage-cost, which of course implies (if wages are uniform) that price ratios will be equal to embodied labour ratios.[20] What this proposition amounts to, of course, is an affirmation of the truth of the Smithian, Ricardian and Marxian proposition that in the 'early and rude state of society', where there is no profit, the classical 'law of value' acts, as it were, directly, so that price ratios will in equilibrium be equal to embodied labour ratios.

Now, Smith, Ricardo and Marx, having established this proposition, went on to argue that in a capitalist society, where the net product was shared between wages and profits, prices no longer followed this simple rule. The 'law of value' which originally operated in this direct and simple way was now subject, as Ricardo put it, to important 'modifications'.[21] Sraffa, like his classical predecessors, now goes on to consider the nature and causes of these 'modifications'.

Sraffa's explanation of the basic reason for the emergence of the 'modifications' is substantially the same as that of Ricardo and Marx. 'The key to the movement of relative prices consequent upon a change in the wage', Sraffa writes, 'lies in the inequality of the proportions in which labour and means of production are employed in different industries'.[22] It is useful, I think, to begin by explaining this point in Ricardo's terms. Let us assume that we have an economy which consists of three separate industries, A, B and C, in each of which the proportions in which labour and means of production are combined together are different. In other words, the ratio of the wage-

[20] Suppose that a two-industry economy produces a gross output of 400 quarters of wheat and 25 tons of iron. Let the sum of the inputs at wage-cost in the two industries be £200 and £250 respectively. Since the value of the end-product will in each case equal the sum of its inputs at wage-cost, the price of a ton of iron will be 20 times the price of a quarter of wheat. Let the wage per man be £5. This means that 40 units of direct and indirect labour are required to produce 400 quarters of wheat, and 50 units of direct and indirect labour are required to produce 25 tons of iron. Thus 20 times as much labour is required to produce a ton of iron as is required to produce a quarter of wheat. Thus price ratios are equal to embodied labour ratios.

[21] Smith, broadly speaking, believed that the 'modifications' were so important as to render it necessary to throw out the old 'law of value' and to replace it by what amounted to a 'cost-of-production' theory of value. Ricardo agreed that the 'modifications' were important, but argued that it was still possible to say that relative prices were *mainly* determined by relative quantities of embodied labour (and, what was for him more significant, that *changes* in relative prices were mainly caused by *changes* in relative quantities of embodied labour). Marx also emphasised the importance of the 'modifications', but maintained that the old 'law of value' still *ultimately* and *indirectly* determined prices. Sraffa, as will be shown in the last part of this article, in effect follows Marx's line of approach to this problem.

[22] Sraffa, p. 12.

bill to the value of used-up means of production is different in each industry. Such an economy might be the following:

	Value of Used-up Means of Production		*Wages*		*Price*
A.	800	+	200	=	1000
B.	600	+	400	=	1000
C.	200	+	800	=	1000

Wages, we begin by assuming, absorb the whole of the net product, profits being zero. Under these circumstances, the price of the finished product will in each case be 1000, as indicated in the table.

Now, suppose that a class of capitalists arrives on the scene and shares in the net product along with labour. Wages, let us assume, go down by one-half, and as a result of this profits rise from zero to a level which affords an average rate of, let us say, 25 per cent. on the value of the means of production. (We leave aside for the moment the important question of how far profits will *in fact* rise as a result of this particular wage-reduction: we simply assume that they will rise from zero to an arbitrarily-chosen figure of 25 per cent.) The price of each commodity will now be made up of the value of the means of production employed (which we assume for the moment to remain at its original level), plus the wage-bill (now cut by one-half in each case), plus profit at 25 per cent. on the value of the means of production. The situation will then be as follows:

	Value of Used-up Means of Production		*Wages*		*Profits*		*Price*
A.	800	+	100	+	200	=	1100
B.	600	+	200	+	150	=	950
C.	200	+	400	+	50	=	650

It is clear that under these circumstances the prices of the three commodities would have to change from their original levels. If the price of the product of industry A remained at 1000, that industry would show a sort of 'deficit': it would not be able to pay wages at the given rate and at the same time receive profits at the given rate on its means of production. Similarly, if the prices of the products of industries B and C remained at 1000, these industries would show a sort of 'surplus': they would secure receipts which were more than sufficient to pay wages at the given rate and to earn profits at the given rate on their means of production. Therefore prices would clearly have to alter to the levels indicated in the second table above. The relative prices of the three commodities would change in this case, when the wage changed, simply because the proportions in which labour and the means of production are combined in the three industries are different. If these proportions were the same in each case, it is easy to show that relative prices would not change at all from their previous level.[23]

Now, it *looks* from this example as if we could frame a simple general rule about what happens to prices when wages fall. Could we not say that the price of the product of an industry with a relatively low proportion of labour to means of production, like industry A in our example, would rise when wages fell; and that the price of the product of an industry with a relatively high proportion of labour to means of production, like industries B and C in our example, would fall when wages fell? This is in effect what Ricardo said. But this need not in fact necessarily be so. It certainly *looks* from our example as if the price of the product of industry B, say, is bound to fall. But we have so far assumed, as Ricardo usually did, that the value of the means of production employed in industry B remains the same as it was initially—i.e., at 600—in spite of the fall in wages. But suppose that these means of production employed in industry B were themselves produced by an industry like A in our example, where the proportion of labour to means of production is relatively low. The price of the means of production employed in B would then rise when wages fell, so that the price of the *product* of industry B, instead of falling as it does in our example, might actually rise. Thus the movements in the relative prices of any two products, consequent upon a change in wages, come to depend, as Sraffa puts it, ' not only on the " proportions " of labour to means of production by which they are respectively produced, but also on the " proportions " by which those means have themselves been produced, and also on the " proportions " by which the means of production of those means of production have been produced, and so on '.[24]

Now, we could imagine that an industry existed which represented a sort of ' borderline ' between the ' deficit ' and ' surplus ' industries which we have just distinguished. In such an industry, as Sraffa puts it, ' the proceeds of the wage-reduction would provide exactly what was required for the payment of profits at the general rate '.[25]

[23] Take, for example, a situation in which industry A uses up 400 means of production and pays out 800 in wages; industry B uses up 300 means of production and pays out 600 in wages; and industry C uses up 200 means of production and pays out 400 in wages. The prices of the three products in the initial situation will be 1200, 900 and 600—i.e., they will stand to one another in the ratio 4:3:2. If wages fall by one-half and profits as a result rise from zero to 25 per cent., the prices of the products will become 900, 675 and 450 respectively—i.e., they will still stand to one another in the ratio 4:3:2.

[24] Sraffa, p. 15.

[25] Sraffa, p. 13.

Suppose, for example, that there was an industry which employed labour and means of production in such a proportion that on the basis of the initial prices of the means of production the proceeds of the wage-reduction provided exactly the amount that was required to pay profits at the average rate—instead of something less, as in our industry A, or something more, as in our industries B and C. Suppose further—and this is the vital point—that the means of production which this industry employed were themselves produced by labour and means of production in the same proportion, and so on right down the line. There would be nothing in the conditions of production of such an industry which would make its product rise or fall in value relative to any other commodity when wages rose or fell. And the value of such a commodity relative to the value of its own means of production could not possibly change, since the same ' proportions ' would by hypothesis apply in the case of these means of production, *their* means of production, and so on right down the line. Thus one way of expressing the quality of ' invariance ' which the product of this borderline industry would possess is to say that the ratio of the value of the industry's net product to the value of its means of production would always remain the same whatever change took place in the wage. And it is easy to show that this ratio must be equal to the average rate of profits which would prevail over the economy as a whole if wages were zero[26]—the ' maximum rate of profits ', as Sraffa calls it. Sraffa uses the term ' R ' to refer both to the ratio of the value of the net product of the borderline industry to the value of its means of production, and to the ' maximum rate of profits '. So we have:

$$\frac{\text{Value of net product of borderline industry}}{\text{Value of its means of production}} = \text{`Maximum rate of profits'} = R$$

Having set out in a general way the basic condition of an ' invariant ' industry, Sraffa now proceeds to ask whether an industry fulfilling this condition could in fact be found. No actual industry in the economy is likely to fulfil the requirements; but, Sraffa argues, a mixture of industries, or of bits of industries, would do just as well. His next task, therefore, is to show that it is in fact possible to distil, from any actual economy, a sort of composite industry in which the

[26] If the wage fell to zero, the ratio of the value of the net product of the borderline industry to the value of its means of production would become equivalent to the rate of profits in the borderline industry, and by hypothesis this ratio cannot change. Thus if wages are zero, prices in the rest of the economy must so change as to bring the average rate of profit into equality with the ratio of the value of the net product of the borderline industry to the value of its means of production.

ratio of net product to means of production will remain invariable in the face of any change in the wage. Let us take a simple example of the distillation operation which Sraffa undertakes in order to obtain a composite industry which fulfils this basic condition. Take the economy whose conditions of production in physical terms are as follows:

$$\begin{array}{l}375 \text{ qr. wheat} + 6 \text{ t. iron} \rightarrow 750 \text{ qr. wheat} \\ 300 \text{ qr. wheat} + 24 \text{ t. iron} \rightarrow 40 \text{ t. iron} \\ \overline{675} \qquad\qquad\quad \overline{30}\end{array}$$

The net product of this economy consists of 10 tons of iron plus 75 quarters of wheat. Now, suppose that we separate off *two-thirds* of the wheat industry and *one-half* of the iron industry, and treat the two resultant fractions of these industries as constituting together a sort of composite industry.[27] The conditions of production of this composite industry would be as follows:

$$\begin{array}{l}250 \text{ qr. wheat} + 4 \text{ t. iron} \rightarrow 500 \text{ qr. wheat} \\ 150 \text{ qr. wheat} + 12 \text{ t. iron} \rightarrow 20 \text{ t. iron} \\ \overline{400} \qquad\qquad\quad \overline{16}\end{array}$$

Let us now identify the crucial ratio of net product to means of production in this composite industry. The net product consists of 4 tons of iron plus 100 quarters of wheat; and the means of production consist of 16 tons of iron plus 400 quarters of wheat. Thus the ratio is:

$$\frac{4 \text{ t. iron} + 100 \text{ qr. wheat}}{16 \text{ t. iron} + 400 \text{ qr. wheat}}$$

The numerator and denominator of this ratio, it will be noticed, are made up of quantities of the same commodities combined in the same proportions, which means that we can speak of a ratio between the two sets of commodities without the need to reduce them to the common measure of price. The ratio is of course one-quarter. And it is clear that this ratio would remain the same whatever the *prices* of the two commodities happened to be. The ratio between the two sets of commodities in *price* terms would always be the same as it is in *physical* terms—one-quarter. In other words, even though wages altered and prices subsequently changed, the ratio of the value of the net product of this composite or 'standard' industry to the value of

[27] We leave aside for the moment the question of how the appropriate multiplying fractions are arrived at.

its means of production would necessarily remain unchanged. Thus this industry would fulfil the basic condition of invariance which we have already established.

By what subtle magic has this rather startling result been obtained? We have obtained it because the fractions which we selected as our multipliers were cunningly chosen so that in the reduced-scale system the proportions in which the two commodities are produced (20 : 500) are the same as those in which they enter the aggregate means of production (16 : 400). It is only because the multiplying fractions which we chose were such as to yield us a reduced-scale system possessing this particular property that the numerator and denominator of the ratio of net product to means of production have come to consist of quantities of the same commodities combined in the same proportions, so that the ratio necessarily remains invariant to price changes. Sraffa now proceeds to show very elegantly that there is always a set of multipliers, and never more than one set, which when applied to the industries of any actual economy will rearrange them in the 'right' proportions.

Let us now consider what happens to the rate of profits *in the composite or 'standard' industry* when the wage changes. If we write R (as before) for the ratio of net product to means of production, r for the rate of profits, and w for the proportion of the net product going to wages, the relation between wages and profits in the 'standard' industry can be expressed in the form of the following simple relation:

$$r = \mathrm{R}(1-w)$$

Take as an example the 'standard' industry which we have just considered, where $\mathrm{R}=\frac{1}{4}$. Suppose that three-quarters of the net product (i.e., 3 t. iron + 75 qr. wheat) went to wages, so that the remaining one-quarter (i.e., 1 t. iron + 25 qr. wheat) went to profits. The rate of profits would then be:

$$\frac{1 \text{ t. iron} + 25 \text{ qr. wheat}}{16 \text{ t. iron} + 400 \text{ qr. wheat}} \quad (=1/16)$$

And this rate of profits of 1/16, or $6\frac{1}{4}$ per cent., is clearly given by the expression $r=\mathrm{R}(1-w)$, where $\mathrm{R}=\frac{1}{4}$ and $w=\frac{3}{4}$. What this expression says, in essence, is that the rate of profits *in the 'standard' industry* increases in direct proportion to the total deduction made from the wage, the extent of the increase depending on the value of R.

Now comes the final stage in this highly ingenious and persuasive argument. Sraffa maintains that this relation between wages and profits is not limited to our imaginary 'standard' system, but can also be extended to the actual economic system from which the ‘ standard ’ system has been derived. For the actual system, Sraffa argues, consists of the same basic equations as the ‘ standard ’ system, only in different proportions, so that ‘ once the wage is given, the rate of profits is determined for both systems regardless of the proportions of the equations in either of them ’.[28] Thus, Sraffa concludes, the rate of profits *over the economy as a whole* is determined as soon as we know R (the ratio of net product to means of production in the ‘ standard ’ industry, which is equal to the ‘ maximum rate of profits ’), and w (the proportion of the net product of the ‘ standard ’ industry going to wages). Or, to put the point in another way, when the proportion of the net product of the ‘ standard ’ industry going to wages is given, the average rate of profits over the economy as a whole depends upon the level of R.

In the remainder of his book, Sraffa makes extensive use of this simple relation between wages and profits to elucidate a number of difficult theoretical problems. In one chapter, for example, he analyses the case where commodities are produced with means of production which were themselves produced at different periods in the past (and so on down the line), so that the profit element in the prices of these means of production is different, and asks how the relative values of the commodities will vary with changes in the rate of profits.[29] In the second part of his book, again, he deals with the new problems which arise when we take account of the fact of the existence of items of fixed capital which outlast one use and gradually depreciate in value during the course of their life. What generalisations can be made, he asks, on the basis of the theoretical foundations erected in the earlier part of the book, concerning the path followed by this depreciation? Finally, carrying on with the method of successive approximations in much the same way as his classical predecessors, he brings land into the picture, and erects a more complex system of equations in which, if wages are given, the prices of all commodities, the rate of profits, *and* the rents payable on different qualities of land, are all determined. To the historian of economic thought, one of the most interesting features of these extensions of the basic analysis is the number of old friends who are met with. For example, in the chapter on fixed capital Sraffa makes interesting use of the old classical device, first used by Torrens, of treating what is left of fixed

[28] Sraffa, p. 23.

[29] In this chapter, Sraffa deals with the problem of reducing ‘ constant capital ’ (to use Marx’s terminology) to quantities of labour. He points out, in effect, that the reduction operation can in fact be performed, provided that the labour is *dated* labour, since the dating will affect the rate of profits and therefore the prices of the commodities concerned.

capital at the end of the year as a kind of joint product of the industry in which it is used. Of special importance in these later parts of the book are the distinction which is early established between 'basic' and 'non-basic' products,[30] and the general analysis of joint products.

IV

One very important feature of Sraffa's analysis remains to be commented upon—his implied rehabilitation of the classical labour theory of value in something very like the form which it assumed in the hands of Marx. The Marxian labour theory of value does *not* say, as is vulgarly supposed, that the equilibrium prices of commodities are always proportionate to the quantities of labour required to produce them. It affirms, certainly, that this statement is true of an economy where 'the whole produce of labour belongs to the labourer'; but it agrees—indeed emphasises—that equilibrium prices do *not* normally follow this simple rule in a capitalist economy where part of the net product goes to profits. In a capitalist economy, it is demonstrated, relative prices normally deviate from relative quantities of embodied labour, for reasons which have been described earlier in the present article. Even in a capitalist economy, however, it is argued, the equilibrium prices of commodities can still be shown to be 'indirectly' and 'ultimately' determined by certain crucial ratios of quantities of embodied labour applicable to the economy as a whole. For the deviations of price ratios from embodied labour ratios, given the proportions in which labour and means of production are combined together in each industry, depend upon the level of the average rate of profits; and the level of the average rate of profits, it is claimed, depends in its turn upon the crucial ratios of quantities of embodied labour to which I have just referred. Thus if it can in fact be shown that the average rate of profits is determined by these

[30] A 'basic' product, roughly speaking, is one which enters (no matter whether directly or indirectly) into the production of *all* commodities, and a 'non-basic' product is one which does not. A 'luxury' product, for example, which is not used (whether as an instrument of production or as an article of subsistence) in the production of other products, is 'non-basic'. (See Sraffa, pp. 7-8.) The important feature of 'non-basic' products is that they 'have no part in the determination of the system', their rôle being 'purely passive'. In other words, 'the price of a non-basic product depends on the prices of its means of production, but these do not depend on it', whereas 'in the case of a basic product the prices of its means of production depend on its own price no less than the latter depends on them' (p. 9). Specialists in Marxist theory will note the relevance of this part of Sraffa's analysis to an important question which arose in the course of the debate on the so-called 'transformation problem'—the question (raised in particular by Bortkiewicz) as to whether the conditions of production of luxury goods enter into the determination of the rate of profits.

embodied labour ratios, we can reasonably conclude that the very deviations of equilibrium price ratios from embodied labour ratios are themselves determined by 'quantities of embodied labour'.

Marx's method of showing the dependence of the rate of profits on 'quantities of embodied labour' in this sense can be illustrated with the aid of the following simple model:

	Means of Production	*Wages*	*Surplus Value*
A.	40	160	80
B.	60	90	45
C.	120	80	40

We here assume that the economy consists of three separate industries, A, B and C. The quantities under the three headings 'Means of Production', 'Wages' and 'Surplus Value' are each reckoned in terms of hours of labour. Take industry A as an example. In industry A, the means of production used up during a given period of production are assumed to 'contain' or 'embody' a total of 40 hours of past labour. The total amount of present or direct labour expended in the industry during the period is assumed to be 240 hours—the sum of the figures 160 and 80 under the respective headings 'Wages' and 'Surplus Value'. It is assumed that in two-thirds of this total working time—i.e., 160 hours—the direct labourers are able to contribute just enough value to the product to cover their own wages. In the remaining 80 hours they contribute what Marx called 'surplus value', which he assumed to be the sole source of capitalist profit. The same interpretation is given to the figures for industries B and C, where, it will be noticed, the proportions in which labour and means of production are combined together are different from those in industry A. The ratio of surplus value to wages is assumed to be the same (in this case 1 : 2) in each industry.

The average rate of profits in this economy, Marx argued, can be found by taking the aggregate surplus value yielded over the economy as a whole (165) and redividing it among the three industries in proportion to the means of production employed in each. Or, to put the point in a way which is perhaps easier to understand, the average rate of profits will be determined by the ratio of aggregate surplus value to aggregate means of production. In this case it will clearly be three-quarters, or 75 per cent.[31] This ratio of aggregate

[31] Marx, in common with his classical predecessors, generally assumed that wages were 'advanced' out of capital. This meant that in working out the rate of profits he normally related surplus value to means of production

quantities of embodied labour, then, determines the average rate of profits, and thus the deviations of equilibrium price ratios from embodied labour ratios.

At first sight this analysis might seem to have little in common with Sraffa's. But suppose we go on to postulate, as Marx himself did, an industry in which the ratio of used-up means of production to wages is equal to the ratio of these quantities when they are aggregated over the economy as a whole. Industry B in our illustration is clearly an industry possessing this characteristic—it is an industry in which, to use Marx's terminology, the 'organic composition of capital' is equal to the 'social average'.[32] In such an industry, as can be seen from the illustration, the ratio of surplus value to means of production (45 : 60) is equal to the ratio of these quantities over the economy as a whole (165 : 220). We can thus say, as Marx did,[33] that the average rate of profits over the economy as a whole is determined by the ratio of surplus value to means of production *in this industry B*, whose conditions of production represent a sort of 'social average'. Or, to put the same proposition in another way, the average rate of profits over the economy as a whole is given by the following expression:[34]

$$\frac{\text{Labour embodied in net product of industry B}}{\text{Labour embodied in its means of production}}\left(1-\begin{array}{l}\text{proportion of}\\ \text{net product of}\\ \text{industry B going}\\ \text{to wages}\end{array}\right)$$

Now, the similarity between this Marxian relation and that expressed in Sraffa's $r=R(1-w)$ is surely very striking. For, in the first place, let us note that Sraffa's R, although usually expressed as the ratio of the *value* of the net product of the 'standard' industry to the *value* of its means of production, is in fact equal to the ratio of the *labour embodied* in the net product of the 'standard' industry

plus wages. Following Sraffa's precedent (p. 10), I am assuming here that the wage is not in fact 'advanced', but 'paid *post factum* as a share of the annual product', which means that the rate of profits is obtained by relating surplus value to means of production alone. To drop this particular assumption of Marx's does not affect the essence of his analysis, and greatly facilitates the comparison with Sraffa which is made below.

[32] See *Capital*, Vol. III (Kerr edn.), p. 193. In the example, the 'organic composition of capital' in industry B (60 : 90) is clearly equal to the 'organic composition of capital' over the economy as a whole (220 : 330).

[33] *Capital*, Vol. III, p. 204.

[34] In this expression the 'net product' is taken to consist of wages plus surplus value (as, in effect, it is with Sraffa). Thus the expression is merely another way of formulating the ratio of surplus value to means of production, each of these quantities being estimated in terms of embodied labour.

to the *labour embodied* in its means of production.[35] In other words, Sraffa is postulating precisely the same relation between the average rate of profits *and the conditions of production in his 'standard' industry* as Marx was postulating between the average rate of profits *and the conditions of production in his industry of 'average organic composition of capital'*. What both economists are trying to show, in effect, is that (when wages are given) the average rate of profits, and therefore the deviations of price ratios from embodied labour ratios, are governed by the ratio of direct to indirect labour in the industry whose conditions of production represent a sort of 'average' of those prevailing over the economy as a whole. Marx reached this result by postulating as his 'average' industry one whose 'organic composition of capital' was equal to the 'social average'. But his result could only be a provisional and approximate one, since in reaching it he had abstracted from the effect which a change in the wage would have on the prices of the means of production employed in the 'average' industry.[36] Sraffa shows that the same result can be achieved, without abstracting from this effect at all, if we substitute his 'standard' industry for Marx's industry of 'average organic composition of capital'. Sraffa's 'standard' industry, seen from this point of view, is essentially an attempt to *define* 'average conditions of production' in such a way as to achieve the identical result for which Marx was seeking.

RONALD L. MEEK

University of Glasgow

[35] Cf. Sraffa, pp. 16-17. The reason for the equivalence of the value ratio and the embodied labour ratio is as follows: When profits are zero, the prices of all commodities are proportionate to the quantities of labour required to produce them (as has been shown above, pp. 125-6). And when profits rise *above* zero, the ratio R by hypothesis does not change. Thus whatever the level of profits the value ratio remains equal to the embodied labour ratio.

[36] Marx made this abstraction quite deliberately, and was fully aware that his result was therefore provisional and approximate. See *Capital*, Vol. III, pp. 241-3.

[24]

Jevons's Theory of Capital and Interest*

The principal purpose of this paper is to examine closely the internal logic of Jevons's theory of capital and interest, as expounded in the second edition of his *Theory of Political Economy* ; we shall not be concerned with the sources of this theory, with its influence on later writers or with the changes in Jevons's position as between the *Brief Account*, the first two editions of the *Theory* and the *Fragment on Capital*.[1] By contrast with the discussions of Jevons's theory provided by Keynes, Robbins and Stigler,[2] our discussion will not consider this theory in the light of an alternative theory of capital and interest but will be concerned simply with its internal coherence.

The paper is divided into four main sections. In section I we consider Jevons's concepts of capital, the amount of capital and the amount of investment ; in section II we turn to the average time of investment, which Jevons derives from the amounts of capital and of investment, and consider the limitations to the use of this average period ; Jevons's theory of interest is considered in section III. In section IV we draw attention to the relations between Jevons's theory of interest and certain other components of his thought, the purpose of this section being to stimulate thought about these relations rather than to provide an analysis of their causes and significance.

1. Capital, Amount of Capital and Amount of Investment

Jevons's clearest statement of his views on the nature of capital appears in the section of the *Theory*, Chapter VII, entitled "Free and Invested Capital". He writes :

> "I believe that the clear explanation of the doctrine of capital requires the use of a term *free capital*, which has not been hitherto recognised by economists. By free capital I mean the wages of

*I should like to thank P. Garegnani for his penetrating and constructive criticism of the first two drafts of this paper ; helpful comments on the third draft were received from R. D. C. Black, K. H. Hennings, B. MacLennan and J. S. Metcalfe. I am grateful to the Nuffield Foundation for awarding me a fellowship, during the tenure of which this paper was largely written.

[1]Reference will, of course, be made to the *Brief Account* and to the *Fragment*. It may be noted here that all references to the *Brief Account*, the *Theory* and the *Fragment* will be given by means of the page references which apply to both the fourth and the fifth editions of *The Theory of Political Economy*. Other works will be cited in such a way as to enable the reader to find the full reference in the list of works cited, at the end of this paper.

[2]Keynes, "William Stanley Jevons, 1835–1882."
Robbins, "The Place of Jevons, etc."
Stigler, *Production and Distribution Theories*, chap. 2.

labour, either in its transitory form of money, or its real form of food and other necessaries of life. The ordinary sustenance requisite to support labourers of all ranks when engaged upon their work is really the true form of capital." (pp. 242–3. Jevons's emphasis.)

This statement would seem to raise two questions.

(i) *Capital and the Amount of Capital*

The first problem arises from Jevons's "double" definition of free capital as *either* a sum of money, money wages, *or* a set of heterogeneous commodities, real wages in the form of food and other necessaries ; clearly, free capital cannot be both a single quantity and a set of quantities. In section 21 of the *Brief Account* (pp. 311–12) and at p. 223 of the *Theory*, Jevons refers simply to "maintenance of labourers" and "means of sustenance," in both cases, that is, to real wages as a bundle of heterogeneous commodities, no reference being made to an amount of money. Yet it was not just a careless slip which led Jevons to provide an alternative meaning to "free capital" in the passage quoted above, for on the immediately following page, (p. 244), he turns to consider "the rate of interest for free capital" and is thus obliged to consider capital as a homogeneous quantity. We may interpret the quoted statement as follows. Free capital, for Jevons, consists of a bundle of heterogeneous commodities, the wage commodities ; the proportions in which the commodities enter this bundle must, of course, depend on relative prices.[1] Jevons then makes free capital homogeneous by valuing it at money prices and is thus able to refer to the *amount* of free capital as money wages.[2]

[1]One could regard these proportions as *given*, in a given country at a given time, but *Jevons* could not do this without ignoring his own theory of demand based on utility-maximisation. We may note, however, that Jevons seems to have been tempted to ignore his demand theory when in the *Brief Account* (p. 312) he defined capital as those commodities "supplying a labourer's ordinary wants and desires".

[2]Throughout our discussion of the amounts of capital and of investment, we shall follow Jevons's usual practice of defining these concepts in terms of *value* but it may be useful to consider here the fact that Jevons sometimes seems to refer to amounts of capital and of investment as either *physical* quantities or as amounts of *utility*.

Although Jevons defines free capital as a bundle of heterogeneous commodities, on pp. 233, 234, 248 and 249 he states that capital, interest and wages have the *single* dimension Mass ; on p. 235 he even discusses lending and borrowing in terms of physically specified quantities of an individual commodity. On both pp. 234 and 249–50, deferment of consumption is discussed in terms of a single commodity. In some of these cases, Jevons may have in mind the fact that the *value* of a set of commodities is expressed as a physical mass of the value-standard, e.g., two ounces of gold, but in others it is difficult to avoid the impression that he has slipped away from his own definition of free capital, into thinking in terms of a one-commodity world (see, e.g., the complete paragraph on p. 249).

Footnote continued on page 33

The above interpretation is consistent not only with Jevons's statement that capital consists of maintenance or sustenance (referred to above) but also with the fact that Jevons frequently refers to "amount of capital" as homogeneous wages. Thus, on p. 230 there is a numerical example in which the amount of capital is given in shillings, there are algebraic examples in which the amount of capital appears as "amount of wages" (p. 235) and as the cost of labour (p. 236) and on p. 257 there is a numerical example in which the amount of capital is expressed as so many pounds-sterling. It will have been noticed that in each case in which Jevons makes explicit the units in which the amount of capital is measured, they are money units. Jevons does not discuss money in the *Theory* (or in the *Brief Account*) but since he would clearly not wish the *amount* of free capital, corresponding to a given free capital, to depend on changes in prices in terms of paper-money and since he does elsewhere use money-price to mean price in terms of the produced commodity gold,[1] we may reasonably take "amount of free capital" to mean "value of free capital in terms of gold."

Free capital per unit of labour, then, is the real wage-rate expressed as a bundle of commodities, while the *amount* of free capital, per unit of labour, is simply the wage-rate in terms of gold. It will be clear that neither the physical composition of free capital nor the amount of free capital can be known independently of relative prices and hence of the interest rate.

(ii) *The Amount of Investment*

We may now turn to the second question raised by Jevons's above-quoted statement. Jevons says that "sustenance" is "the true *form* of capital" (my emphasis), thus recalling his earlier statement that,

> "The *current means of sustenance constitute capital in its free or uninvested form*". (pp. 223–4, Jevons's emphasis.)

Why does Jevons refer to free capital, sustenance, as a *form* of capital? A reading of section 21 of the *Brief Account* and of pp. 222–4,

[1]See *Methods of Social Reform*, (1904), p. 110, *Investigations in Currency and Finance*, pp. 20–24, 31–33 and *Primer of Political Economy*, pp. 106–8.

Footnote 2 continued from previous page.

In section 22 of the *Brief Account* (p. 312) and in the *Fragment* (pp. 295, 300–2) amounts of capital and of investment are referred to as amounts of utility and as amounts of utility multiplied by time; this notion also appears, less explicitly, in "Dimensions of Capital, Credit and Debit". (*Theory*, pp. 233–5). It is not easy to render this notion coherent within *Jevons's* theory, since Jevons himself argued that the marginal utility of income, and therefore the marginal utility of any commodity, is different as between different individuals (see pp. 139–41). Thus even if we knew which commodities were to be "valued" in utility, we should not know what marginal-utility weights to use in the "valuation". It is thus difficult to see why Keynes regarded as "admirable" those passages in which Jevons measured capital in terms of utility, ("William Stanley Jevons 1835–1882," p. 534).

242–4 of the *Theory* strongly suggests that Jevons calls free capital the "true form of capital" in order to stress the relationship between his concept of capital as maintenance and the then more common conception of capital as consisting of not only maintenance of labourers but also tools, buildings, semi-finished products, etc. Jevons, as it were, "resolves" the tools, buildings, etc., into quantities of maintenance invested at various points of time in the past. Thus in the *Brief Account* he writes,

> "Capital, in short, is nothing but *maintenance of labourers*. It is, of course, perfectly true that buildings, tools, materials, etc., are a necessary means of production ; but they are already the product of labour assisted by capital or maintenance". (p. 312, Jevons's emphasis),

while in the *Theory* we read,

> ". . . I would not say that a railway *is fixed capital*, but that *capital is fixed in the railway*. The capital is not the railway, but the food of those who made the railway".[1] (p. 243, Jevons's emphasis.)[2]

It is Jevons's conceptual resolution of produced means of production into past investments of free capital that leads him to so emphasise both the distinction and the relation between *amount of capital* and *amount of investment*.[3] The need for a distinction is clear; two produced means of production, each of which can be "resolved" into past investments of free capital totalling £1000, will not be economically equivalent, if for one of them the £1000 are all invested ten years before it is used up in production, while for the other they are all invested only one year before. Less immediately clear is what concept can best reflect the time-series of amounts of free capital into which a means of production can be resolved or, in other words, what is the appropriate definition of the "amount of investment". Jevons appears to have no doubt on the matter ; he writes,

> "The amount of investment of capital will evidently be determined by multiplying each portion of capital invested at any moment by the length of time for which it remains invested." (pp. 229–30).

It becomes clear from the subsequent discussion that to find the amount of investment when the capital involved is not all invested at the same moment of time, we are to sum the various terms, each of the

[1] It is clear from the immediately following sentence that Jevons here used "food" as shorthand for "food and other necessaries".

[2] See also *Primer of Political Economy*, p. 46.

[3] Or, more fully, the "amount of (free) capital invested" and the "amount of investment of capital".

form "amount of free capital $\times$ time". Why does Jevons say that the amount of investment is "evidently" to be defined in this way? As we shall see below, the amount of investment so defined is relevant for a theory of *simple interest* but we can see no other use for Jevons's amount of investment.

It may be noted that Jevons's amount of investment may be thought of as the number of units of labour which could be currently employed, at the current wage, *if* the actual means of production were to be replaced by means of subsistence of the same value and indeed Wicksell referred to the Jevonian concept of capital as a fund of subsistence, as a sum of *potential* wages.[1] In fact the means of production cannot be so replaced, at a given moment in time, but this way of viewing the amount of investment helps us to understand Jevons's "reduction" of capital to maintenance and to see why he called real wages "free capital". The means of production are conceived as equal in value to a sum of (stored up) real wages which, while not available now as real wages, becomes available over time.[2]

Like the amount of capital, the amount of investment cannot be known independently of relative prices and hence of the interest rate. Jevons, in his theory of interest, does not, however, relate the rate of interest directly to the amount of investment, itself a function of the interest rate, but relates it rather to the average period of production. To this concept we now turn.

II. The Average Period of Production and its Use

(i) *The Average Period of Production*[3]

On pages 229 and 230 of the *Theory* Jevons defines the amount of investment. He proceeds immediately on p. 231 to his well-known diagrammatic analysis of agricultural production in which labour, applied at a constant rate over one year, produces a harvest which is consumed at a constant rate over the following year. After stating that the amount of investment will be given by the area of his famous "triangle," Jevons writes,

[1] *Lectures*, vol. I, p. 168.

[2] Cf. Wicksell, *Value, Capital and Rent*, p. 146.

[3] The reader is reminded that *Jevons* did not use this expression and that the periods of time to which he does refer often include the time required for the using up of a commodity, as well as that required for its production. Cf. Schumpeter, *History of Economic Analysis*, pp. 903, 908 and Stigler, *Production and Distribution Theories*, p. 24. The distinction between investment and production periods is not important for our discussion, since we argue that Jevons's theory is inadequate even in the simple case in which the two periods coincide.

> "Now the area of a triangle is equal to the height multiplied by half the base ; and as the height represents the greatest amount invested, . . ., half the base, or one year, is the *average time of investment of the whole amount.*" (p. 231, Jevons's emphasis).

The same definition of the average time of investment is implicit in the rest of the paragraph from which this quotation is taken (pp. 231–2).[1] It will be clear, in this agricultural example at least, that Jevons defines the relevant concepts in such a way that The Average Time of Investment equals The Amount of Investment of Capital *divided by* The Amount of Capital Invested.

The need for the qualification "in this agricultural example at least", arises from the fact that Jevons nowhere else gives an explicit definition of the average period, while the agricultural example is described as a "simple illustration" (p. 231), so that we are given no explicit answer to the question whether Jevons would use the same definition of the average period in more complex cases. Nor is an answer provided by the three numerical examples given by Jevons on pp. 226 and 228 since these examples should probably be interpreted as being of the same kind as the agricultural example. i.e., as exhibiting linear investment/disinvestment functions.

Since both the amount of investment and the amount of capital can be defined for the aggregate of commodities, just as for an individual commodity, we could define a social average period as the ratio of the aggregate amount of investment to the aggregate amount of capital. And at one point Jevons appears to have such a conception in mind, for he refers to,

> "whatever improvements in the supply of commodities lengthen[2] the average interval between the moment when labour is exerted and its ultimate result or purpose is accomplished. . . ." (pp. 228–9, italicised in original).

Since "commodities" appear in the plural but "the average interval" only in the singular, it seems reasonable to interpret *this* average period as a social average over all the commodities in question.

As noted above, the average period for a *given* technique, may be written as the ratio between the amount of investment and the amount of capital. This draws our attention to the fact that the average period

[1]That, under the conditions here assumed by Jevons, the amount of investment depends *only* on the total time period and the greatest amount invested (being completely independent of the time profile of investment) raises, in a very sharp form, the question "What, then, is the significance of the amount of investment?" As stated above, we can see no answer to this question other than that simple interest is to be assumed.

[2]In an editorial footnote (p. 228), H. S. Jevons suggests that "lengthen" should be read as "involve a lengthening of".

depends only on the conditions of production and is thus defined independently of income distribution. Suppose, for example, that the production of a certain commodity by means of a given technique requires a labour input of l_2 units two time periods before the commodity becomes available for instantaneous consumption and of l_1 units one time period before. If the wage rate is w, then the amount of capital involved will be (wl_2+wl_1), while the amount of investment will be given by

$$K = 2wl_2+wl_1.$$

The average period, T, is thus

$$T = \left(\frac{2wl_2+wl_1}{wl_2+wl_1}\right) = \left(\frac{2l_2+l_1}{l_2+l_1}\right),$$

which is completely defined by the conditions of production.

As with the amount of investment from which it is derived, we can see no role for the average period if it is not to be used in a theory of simple interest.[1]

(ii) *Compound Interest and Fixed Capital*

It has been asserted above that neither the amount of investment nor the average period of investment derived from it are useful concepts, unless they are to be used in a theory of interest in which simple interest is assumed. The purpose of this section is to show why these concepts are of no use if compound interest is assumed and to introduce another restriction on their use, which has so far been glossed over, i.e., the requirement that all produced means of production should be of a circulating, and not of a fixed, nature. Since an ample discussion of these points is provided by Garegnani,[2] they will be dealt with here only by means of simple examples.

Consider first a single commodity, the production of which requires the use of no fixed capital. Two years before the commodity emerges, ready for instantaneous consumption, one unit of labour is applied and another one unit is applied one year before the commodity becomes available. Let w be the wage-rate. We then have immediately that

$$K = 2w+w = 3w$$

and thus

$$T = \frac{K}{2w} = \frac{3}{2} \tag{1}$$

[1]The concept of the average period is not mentioned in the *Brief Account* and is referred to only in passing (p. 299) in the *Fragment*.

[2]*Il Capitale, ecc.* pp. 38–9, 61–5.

Now the value of the commodity, assuming simple interest, is clearly given by

$$\begin{aligned} V &= w(1+2r)+w(1+r) \\ &= 2w+3wr \\ &= 2w\left(1+\frac{3}{2}r\right) \qquad (2) \end{aligned}$$

A meaningful average period, t, must satisfy the relation

$$V = 2w(1+rt) \qquad (2')$$

and, as is obvious from (1), (2) and (2′), Jevons's T does indeed satisfy both (2) and (2′). Since, as mentioned above, T is independent of r, the way is now open to express r as a known function of variables independent of r.[1] With compound interest, however, a meaningful average period, t, must satisfy

$$V = w(1+r)^2+w(1+r) = 2w(1+r)^t. \qquad (2'')$$

Now $t = \frac{3}{2}$ does *not* satisfy (2″), unless r is zero, and nor indeed is it possible to find any other fixed value of t to satisfy (2″), since this relation defines t as an increasing function of r. It thus becomes impossible to determine r as the marginal product of a technically defined period of production.

Now consider the production of a machine (fixed capital) and, to emphasise that we are now considering an additional restriction on the relevance of the period of production, suppose that interest is not compounded. Let one unit of labour create instantaneously a machine, which after one year suddenly produces a product of value $\left(\frac{V}{2}\right)$ and after two years suddenly produces another output of value $\left(\frac{V}{2}\right)$ and disintegrates. We must have that

$$w = \left(\frac{V}{2}\right)\left(\frac{1}{1+r}\right) + \left(\frac{V}{2}\right)\left(\frac{1}{1+2r}\right)$$

or

$$V = 2w\left[\frac{(1+r)(1+2r)}{(2+3r)}\right] \qquad (3)$$

Now a meaningful average period of investment, t, must satisfy

$$V = w(1+rt) \qquad (4)$$

[1]Cf. Garegnani, *Il Capitale, ecc.*, Part I.

and from (3) and (4) we obtain

$$t = \left(\frac{3+4r}{2+3r}\right).$$

t depends on r and only for r equal to zero does $t = \frac{3}{2}$, which is the value of Jevons's average period of investment.[1] Thus Jevons's average

[1] $K = w+\frac{1}{2}w = \left(\frac{3}{2}\right)w\,;\ T = \frac{K}{w} = \frac{3}{2}.$

period is of no significance for interest theory when fixed capital is involved, *even if we assume simple interest*; the same conclusion naturally applies also to the amount of investment. We may now ask whether Jevons did restrict the use of his concepts of amount of investment and average period of investment to situations of simple interest and no fixed capital.

(iii) *Fixed Capital*

It will have been noticed that in our example of the way in which fixed capital makes the average period irrelevant, the "fixedness" of the capital consists not in its calendar durability, that is only two years, but in the fact that the machine yields output at more than one point of time. Given this meaning of "fixedness", it perhaps hardly needs to be demonstrated that Jevons does write of the amount of investment and the average period of investment in contexts which involve the existence of fixed capital. In the section designed to make "our notions of the subject still more exact and general", (p. 232), the amount of investment is defined on the express assumption that the product will be yielded up at various points of time (p. 233). The spade, plough and cotton-mill examples (pp. 226, 228) are all examples of fixed capital and yet in each case an average period is given (and independent of the rate of interest), while on p. 246 it is stated that

> "Every new *machine* or other great invention will usually require a fixation of capital for a certain *average time*. . . ." (my emphases).

In the section "Fixed and Circulating Capital" Jevons writes,

> ". . . we must say that no precise line can be drawn between the two kinds. The difference is one of amount and degree. The duration of capital may vary from a day to several hundred years; the most circulating is the least durable; the most fixed the most durable." (p. 242)

As we have seen, a precise *qualitative* distinction *can* be drawn between fixed and circulating capital, it has nothing to do with the durability of capital[1] and it is crucial for the relevance of Jevons's average period of investment. There is thus a connection between Jevons's attitude to the circulating/fixed distinction and his frequent reference to amounts of investment and average periods of investment in contexts which render these concepts inappropriate.

(iv) *Compound Interest*

It is less clear whether Jevons attempted to use his two concepts in connection with compound interest, since most of his explicit references to either simple or compound interest are not related to these concepts.[2] There are, however, two indications that Jevons regarded his amount of investment as inappropriate to compound interest. Thus in his discussion of the housebuilding example, in which he considers the effect of varying the time period over which a given amount of wages is paid out, Jevons writes, "Thus when the whole expenditure is ultimately the same, the amount of investment is simply proportional to the time. The result would be the more serious if the accumulation of compound interest during the time were taken into account ; but the consideration of compound interest would render the formulae very complex, and is not requisite for the purpose in view". (p. 236). Jevons is, then, aware that his amount of investment is inappropriate with compound interest but here, as elsewhere, he neither states why one should be interested in the case of simple interest nor defines the amount of investment appropriate to compound interest. Jevons *may* again be pointing out that his amount of investment does not relate to compound interest when in section 25 of the *Brief Account* (pp. 313–4), he repeatedly says that a one year increase in the absolute period will increase the amount of investment by "at least" the increment given by his formula ; no hint is given as to how the true increases should be calculated.

III. Capital, the Period of Production and Interest

(i) *The Derivation of the "General Expression"*

Let us first consider the argument in the section "General Expression for the Rate of Interest", *as if* Jevons intended it to apply only to a single-commodity economy ; as will be shown below, he certainly did not so intend it and this must be borne in mind throughout the present section.

[1] In *Il Capitale, ecc.* p. 23, n. 6, Garegnani provides the following example : ". . . the must which requires ten years to mature into a given kind of wine is circulating capital ; a machine which wears out after six months use is fixed capital", (my translation).

[2] The section "General Expression for the Rate of Interest" will be considered separately below, in section III.

Jevons considers a point-input, point-output production process and after introducing his notation for the increment of produce which results from an increase in the absolute period of production he writes,

> "The ratio which this increment [of produce] bears to the increment of investment of capital will determine the rate of interest." (p. 245).

The argument then proceeds as follows :

> "Now, at the end of the time t, we might receive the product Ft, and this is the amount of capital which remains invested when we extend the time by Δt. Hence the amount of increased investment of capital is $\Delta t.Ft$." (pp. 245–6).

As we have seen above, the only explicit definition of the amount of investment ever given by Jevons, is such as to entail that in the present case, if the commodity wage for the given amount of labour (p. 245) be w, the amount of investment increases from

$$\text{“}wt\text{” to “}(w+\Delta w)(t+\Delta t)\text{”}$$

when the absolute period is increased from t to $(t+\Delta t)$. Since the difference between these two amounts cannot be equal to $\Delta t.Ft$, when the rate of interest is positive, it follows that either Jevons is making a simple mistake or he is abandoning his only explicit definition of the amount of investment. Since we know from the house-building example (p. 236) that Jevons was ready to abandon his definition when compound interest was involved and from Wicksell's analysis of maturing wine[1] that Jevons's formula is correct when compound interest is allowed for, it seems reasonable to assume that Jevons was being unfaithful to his definition rather than misapplying it.[2]

Thus as soon as he considers compound interest, Jevons has to abandon the only "amount of investment" which he ever defines explicitly ; why then did he bother to define it and, indeed, to so stress its importance? The only answer we can suggest is that, in his pioneering work, he was less than completely clear that the "amount of investment" relevant for compound interest is qualitatively different from that relevant for simple interest ; it cannot be defined independently of the rate of interest. This, in turn, means that it is no longer possible to define a technical "average period" which is independent of income distribution. The suggestion that Jevons failed to realise the difficulty of transferring the concepts of amount of investment and

[1] *Lectures*, vol. I, pp. 172–181.

[2] On the other hand, Jevons *does* misapply his "marginal product" formula for the rate of interest (p. 245), since $\Delta t.Ft$ is the "value of the increment of investment" and *not* the "increment of the value of investment", the latter being what he needs.

average period to flow-input, flow-output contexts with compound interest is strongly supported by the evidence presented in section (ii) below.

(ii) *The Use of the General Expression*

That Jevons did not intend his formula to apply only to a one-commodity world is proved, if proof be needed, by the opening sentence of the relevant section,

> "We may obtain a general expression for the rate of interest yielded by capital *in any employment* provided that. . . ." (p. 245, my emphasis),

and by the fact that the preceding section is concerned with the uniformity of interest "in all employments" (p. 244) and in "every trade" (p. 245). The question therefore arises as to whether the formula is to hold in *each* employment separately or only to the aggregate of industries, i.e., to produce as a whole. The first, or "industry by industry" interpretation is strongly supported by the fact that in the later section "Advantage of Capital to Industry", Jevons uses his formula for the rate of interest while discussing "any branch of industry", (p. 257).[1]

There will be no need to emphasise that the point-input, point-output production process, on the basis of which Jevons derives his formula, is a very special case.[2] Of more interest are the fact that Jevons used his formula as if it were *not* restricted to this very special case and the resulting question how he came to do this ; this question is particularly pressing since Jevons admitted the correctness of another formula derived "from the supposition of different conditions".[3]

The very phrase "general expression for the rate of interest" must surely have made Jevons aware of the question in what sense a result obtained from a special case can be regarded as "general". Unless Jevons used the word general in a very special way, he must have been claiming that the special case considered was only considered in order to simplify the *derivation* of the result, this result being of more general

[1]Jevons actually identifies the rate of interest with $F'(t)$ rather than $\frac{1}{F}F'(t)$, (p. 258) ,but this is clearly just a slip.

[2]Cf., for example, Wicksteed, *Commonsense*, vol. II, p. 753, Stigler, *Production and Distribution Theories*, p. 29, Black, Editorial Introduction to *Theory* (Penguin ed.) p. 29. We should not follow Stigler (pp. 41–43) in over-estimating Wicksteed's "improvement" of Jevons's analysis. Wicksteed completely glosses over the problem of how capital and product can be measured on a consistent basis, in effect assuming that they consist of the same homogeneous commodity, i.e., assuming away most of the problems of capital and interest (see *Commonsense*, pp. 748–50).

[3]Letter to G. H. Darwin, *Letters and Journal*, p. 301.

validity. Jevons's subsequent use of his formula is entirely consistent with his having intended this implicit claim. In the paragraph immediately following the derivation of his formula, Jevons shows no hesitation in relating his formula to the (supposed) fact that

> "Every new machine or other great invention will usually require a fixation of capital for a certain average time" (p. 246),

a fact that obviously cannot be fitted into the point-input, point-output mould. In other words, Jevons is quite ready to apply his formula to flow-output cases, despite the problem of how to interpret "produce" in this case, and to do so by reinterpreting the t of his formula as an "average time". He does not appear to notice that this average time, to be significant, must be a function of the rate of interest (which he claims to be determining, through his formula, as a known function of t). In the section "Tendency of Profits to a Minimum", Jevons states that the doctrine of the historical fall of the interest rate "is in striking agreement with the result of the somewhat abstract analytical investigation given above. Our formula for the rate of interest shows that . . ." (pp. 253–4) ; Jevons regards his formula as "somewhat abstract" but as nevertheless immediately relevant for the interpretation and explanation of broad historical trends in a real economy with many commodities, produced (presumably) by complex processes. Again, as mentioned above, Jevons uses his formula in discussing "any branch of industry" (pp. 257–8) without placing any restriction on the nature of the production process involved, and in Chapter VIII he uses it for determining the rate of interest in the real-world economy (p. 270). We may note finally the following statement made by Jevons, near the end of his life, in a letter to Edgeworth,

> "I now see that *the whole theory of the matter* is implied in the expression for the rate of interest as given on p. 266 of my 2nd edition", (my emphasis).[1]

Thus both Jevons's practice and his explicit statement suggest that he considered his formula to be of general applicability, despite the fact that he derived it from the analysis of what he knew to be a special case. What led Jevons to adopt such a position? *Perhaps* it was the fact that the formula so clearly focusses attention on the role of time, which was what Jevons wished to emphasise, in combination with Jevons's ability to slide so easily from one "amount of investment" to another and from the absolute period of production to an average period of production, without any recognition that these slippery steps involved important conceptual problems. His assumption was perhaps that what is true in a special case must also be true in the general case,

[1] *Letters and Journal*, p. 439. Note, however, the rather vague word "implied".

provided that we reinterpret the terms of the special result as appropriate "averages" of the corresponding terms of the general case. This assumption is, of course, quite unfounded but is perhaps understandable in a pioneer who thinks that he has grasped the key factor in a problem, time in the case of Jevons's interest theory, and is thus satisfied that no detailed elaboration will disturb the soundness of his essential insight.[1]

(iii) *On Certain Relationships*

On pp. 246–7 of the *Theory*, Jevons clearly suggests the existence of an inverse relation between the rate of interest and some average period. He also asserts the existence of a positive relation between capital per man[2] (that the quantity of labour is fixed is mentioned on p. 245) and some average period. This latter assertion, given the obviously implicit qualification that it is capital *per man* which is in question, also appears at pp. 228–9 and is implied at p. 254. These two relationships imply, of course, the existence of a negative relation between capital per man and the rate of interest. This third relation is asserted in section 25 of the *Brief Account* and is repeated in the *Theory* at pp. 245–6 and at pp. 257–8 , it is almost certainly intended, furthermore, throughout the section "Tendency of Profits to a Minimum".[3]

As is well-known, Jevons was not entitled to assert the existence of these relationships, even for such a simple case as a flow-input, point-output economy, with zero growth rate and net output consisting of only one commodity ; as we have seen, Jevons sought to apply the supposed relationships to far more complex cases. The following two simple examples will suffice to demonstrate that the relations need not hold either for a given technique or, more importantly, across a switch of techniques.

Consider first a one-technique, one consumption commodity economy which is in a stationary state. To produce one unit of the consumption commodity, one unit of labour must be applied ten years in advance and ten units must be applied one year in advance of the appearance of the commodity. With an annual rate of interest r, the

[1]That Jevons was able to maintain his faith in other unestablished theories may be seen from *Letters and Journal*, pp. 350, 392, 394.

[2]Since the phrase "amount of investment of capital per man" is rather cumbersome and since Jevons sometimes refers to "capital" when meaning "amount of investment" (cf. section 25 of *Brief Account*), we shall, throughout this and following sections, use "capital per man" to mean "amount of investment of capital, expressed in terms of gold, per man". This "capital per man" is not to be confused with Jevons's "amount of capital" per man ; the latter is simply the gold wage per man.

[3]"Almost" because Jevons refers only to "accumulation of capital", leaving it implicit that capital *per man* is rising.

commodity wage, w, will be given by

$$w[(1+r)^{10}+10(1+r)] = 1.$$

Now capital per man (in terms of the consumption commodity), k, must satisfy

$$w+rk = (1/11)$$

and this equation, together with the previous one, allows us to write k as a function of r. It is then easy to show that k *rises* as r rises from zero to approximately 25% and that only for higher rates of interest is there an inverse k/r relation.

Now consider an economy in which there are two possible methods of producing the one consumption commodity of which net output consists. For the production of one unit of the consumption commodity, the two processes, A and B, require that the following amounts of labour be expended three, two and one year(s) before the commodity is available :

Years	3	2	1	T
A	1	3·3	1	2·00
B	2	1	2·32	1·94

The final column, headed T, shows for each method the simple Böhm-Bawerkian average period, i.e., the only average period which Jevons defines explicitly. Assuming compound interest, it is easy to show that method A will be used for annual rates of interest, r, such that $0 \leqslant r < 10\%$ *and* $20\% < r$, while method B will be used for $10\% < r < 20\%$: at 10% and 20% interest rates, both techniques may be used. It follows that across the first switch ($r = 10\%$) the average period T is inversely related to r but that across the second ($r = 20\%$) they are *positively* related ; furthermore, the fact of "reswitching" means that *no* other physically defined average period can be inversely related to r for all values of the interest rate. It can also be shown that the value of capital per man, in terms of the consumption commodity, is inversely related to r across the first switch but *positively* related across the second. It follows that, in this example, capital per man is positively related to T across both switches ; whether it is so related to the average period which Jevons would have had us use in connection with compound interest is, of course, impossible to say since he does not define this period.

It may be of interest to note that if we had assumed *simple* interest in the example of the last paragraph, then we should have found a unique switch of techniques at $r = 7{\cdot}14\%$, with both T and capital per man being inversely related to r across this switch. Notice how deceptive would be the argument that at an interest rate of 7%, with a period of only three years, it is not worthwhile to allow for the compounding of interest; in fact, the two kinds of interest give *qualitatively* different results, the small quantitative difference involved being of no significance.[1]

Thus even in an economy producing only one commodity and not using any durable "capital goods" there is *no* basis for Jevons's assertion that the interest rate is inversely related to both the average period and capital per man, while his assertion that capital per man and the average period are positively related cannot be assessed, since the latter term is undefined.[2]

Thus Jevons failed to establish the existence of the crucial relationships upon which his theory of interest turned. It is this fact, and not "unrealistic deficiencies in the first approximation",[3] which lays Jevons's theory open to criticism.

(iv) *Closing the System*

Jevons did not, of course, wish merely to assert the existence of certain economic relationships; these relationships were intended to form part of a closed system of relations which would determine the rate of interest, wages, etc. It will be argued in this section that Jevons sought to close his analytical system by taking capital per man as given and that his system did not in fact explain the interest rate.

Throughout the section "Tendency of Profits to a Minimum" capital is assumed to be increasing but at any given time is used as the determinant of the interest rate; we thus find such phrases as "the rate of interest is high, simply *because* the want of security prevents the due supply of capital", "interest is high, *because* there is not sufficient capital" and "the rate of interest is generally lower *because* there is an abundance of capital" (p. 255, my emphases). The amount of capital per man is again used as the exogeneous explanatory variable throughout the section "Advantage of Capital to Industry"[4] and

[1] At 7% over three years, the simple interest on £100 would be £21, the compound interest £22.50. Wicksell appears to have been misled by such an argument; see *Selected Papers*, p. 179, n. 1.

[2] It is curious that Stigler should be so indignant about this last assertion, when he is unable to know what the assertion is supposed to be. Cf. *Production and Distribution Theories*, p. 26.

[3] Cf. Robbins, *The Place of Jevons*, p. 11.

[4] The discussion of this section relates to an industry, not the economy, but Jevons was probably using the idea of a "typical" industry to illustrate his ideas, since it would otherwise make no sense to take as given the capital available to one industry.

Jevons's theory of wage determination, in Chapter 8, is only comprehensible if the supply of capital is taken as given. It may also be noted that a very early sketch of his theory, in a letter of 1860, already suggests that Jevons regarded the amount of capital as a given, explanatory variable.[1]

If we are correct in suggesting that Jevons closed his analytical system by assuming as given the level of capital per man then two observations are in order. The first is that even if capital per man *could* be taken as given, this would not necessarily determine a unique value of the rate of interest since, as we have seen above, there is no basis within Jevons's theory for asserting the existence of a downward sloping "demand curve" relating capital per man to the rate of interest. A horizontal (or vertical) "supply curve" could thus intersect the "demand curve" any number of times.[2] Thus little attention need be paid to H. S. Jevons's statement that,

> "The examples on p. 255 alone seem to me sufficient to show that he [Jevons] did not, as seems to be suggested by Marshall [in his *Academy* review], overlook the importance of supply in determining market rate of interest. . . . Probably he tacitly assumed that supply of capital remained constant", (p. 280)

if, as seems probable, his statement was intended to mean that his father's interest theory was essentially sound. (It will be clear that this first observation would in any case retain its full force if the *given supply* of capital were replaced by a given "*supply curve*").

The second observation is that it is not self-evident that any meaning can be attached to the assumption that the value of capital is given. The statement that the supply of labour is given has a clear, physical significance; the statement that an "aggregate value" is given, is a statement to which it is difficult to give substantive content.[3] If we attempt to think of the value of capital as the value of a given stock of heterogeneous capital goods then this value will not, in general, be a known constant but rather a known function of the interest rate, the properties of which will depend upon the standard of value chosen.

[1]*Letters and Journal*, pp. 155–6.

[2]We include here the possibility of there being *no* intersection.

[3]It is also difficult to see why one should choose gold as the standard of value in terms of which total capital is to be "given" and why it should be "given" in this standard yet not in others. Since Jevons conceived of capital as stored-up real wages, it might be suggested that the composite wage commodity be taken as the standard; it should be noted that, in Jevons's theory, the physical composition of this "standard" would itself change with changes in the rate of interest and hence in prices.

There is no reason why this function of the interest rate[1] should have a unique stable intersection with the "demand curve" for capital, for positive values of capital and the interest rate ; there may be many intersections or there may be no intersection.

Jevons's theory provides no explanation of the level of the rate of interest ; it follows that it fails to determine wages, prices or quantities. Since Jevons considers an economy in which capital consists of many commodities, there is no possibility of disconnecting his theory of capital and interest from his theory of prices, exchange and consumption[2] and thus the failure of his interest theory results in his theory leaving everything undetermined.

IV. Jevons's Views on Income Distribution

On a number of related questions, Jevons held views which were consistent not only with one another but also with his belief that he had a satisfactory theory of income distribution. Since we have found that belief to be unjustified, it may be appropriate to draw attention to this network of related views and thus, implicitly, to suggest, though not to answer, a number of interesting questions concerning Jevons's ideas.

In 1860 Jevons was already writing that "The common law is that demand and supply of labour and capital determine the division between wages and profits"[3] and this position he never abandoned.[4] He was thus able to refer to "the natural laws which govern the relations of capital and labour, and define inexorably the rates of profits and wages. . . ."[5] Considering the distribution of income to be determined by "natural laws", Jevons quite consistently denounced all trade union activity intended to affect wages as futile and wasteful activity[6] and

[1]This function must not be confused with a Marshallian "supply of capital" curve.

[2]Jevons opened his Chapter VII with the statement that, "In considering the nature and principles of capital, we enter a distinct branch of our subject." If, by this statement, Jevons wished simply to emphasise that the consideration of capital creates *additional* theoretical problems, then one might perhaps concur ; if he implied that his theory of capital is *independent* of the other branches of his theory (and vice-versa), then his opening statement was false.

[3]*Letters and Journal*, p. 155.

[4]See the Preface to the second edition of the *Theory*, the *Theory*, pp. 273–5 and the *Primer of Political Economy*, p. 56.

[5]*The Importance of Diffusing a Knowledge of Political Economy*, p. 20. For further references to the determination of distribution by "natural laws", see *Importance, etc.*, pp. 24, 25, 28, 29 and *Primer*, p. 49.

[6]See the two papers *Trades Societies, etc.*, and *On Industrial Partnerships*, reprinted in *Methods, etc.* ; *The State in Relation to Labour*, pp. 96–98, *The Importance, etc.*, and *Primer*, Chap. VIII.

stated that in doing this he was in no way adopting a political position, let alone reflecting the views of capitalists.[1]

Jevons's theory of income distribution and his views concerning strikes designed to affect wages were, in turn, fully consistent with his "harmonistic" view of his society, a view which can be traced in several of his writings. The *Brief Account*, for example, closes with the statement that "rent, interest, profit, insurance and taxation . . . are so many payments which the labourer makes for advantages enjoyed", (p. 314), a statement which clearly implies the mutually beneficial nature of social relations. Jevons's harmonistic view of his society was not disturbed by his observations of actual conflict since the latter was merely an unfortunate mistake—". . . we only need to throw aside some old but groundless prejudices, in order to heal the discords of capital and labour. . . ."[2] Indeed, as Jevons urged near the end of his life, ". . . the supposed conflict of labour with capital is a delusion".[3]

In the *Theory* we do not find any overt discussion of social harmony but we do find that the section "Relation of Wages and Profits" contains two of the very few moralistic statements to be found in the *Theory*. Thus we read that,

> ". . . the competition to obtain proper workmen will strongly tend to secure to the latter all their *legitimate share* in the ultimate produce." (p. 272, my emphasis),

and that

> "Every labourer ultimately receives the due value of his produce after paying a *proper fraction* to the capitalist for the remuneration of abstinence and risk." (p. 273, my emphasis).

It will be clear that the phrases "legitimate share" and "proper fraction" suggest a harmonistic interpretation of income distribution, even if it is less clear how Jevons came to attribute ethical significance to a distribution of income determined by inexorable natural laws.

In Ricardo's theory of the rate of profits,[4] output per worker and wage per worker[5] are conceived as essentially independent quantities, and the wage is taken as given for the purpose of explaining profits, being determined by social and historical factors extraneous to the

[1]See *Importance, etc.* and *Methods, etc.* (1833), p. 102.

[2]*Methods, etc.* (1904), p. 143.

[3]*State in Relation to Labour*, p. 98.

[4]It would be unnatural to refer to the rate of *interest* when discussing Ricardo; it is for this reason that we here refer to the rate of profit and not because we wish to emphasise any distinction between interest and profit.

[5]We refer here to the wage expressed as a bundle of commodities, its value, at the "natural" prices of the commodities, being the "natural" wage, (cf. Ricardo's *Principles*, Chap. V, *passim*).

theory of value and distribution.[1] Emphasis was thus placed upon the inverse relation between wages and the rate of profit and it was suggested that wages are not inexorably determined. It was thus possible to use Ricardo's theory to support a "conflict" rather than a "harmony" view of capitalist society and to provide a theoretical justification for workers' action to affect wages. While Ricardo did not make such use of his theory, others did.[2]

Since Ricardo's approach to the explanation of income distribution was so fundamentally different from his own, Jevons quite naturally wished "to fling aside, once and for ever" what he regarded as "the mazy and preposterous assumptions of the Ricardian school".[3] His hostility to the Ricardian theory, his hostility to the workers' wages struggle and his harmonistic view of his society, were consistent with each other and with the explanation of interest which he sought to provide in his theory.[4] As we have seen, that theory is unacceptable on purely logical grounds.

V. Conclusion

We have asserted the following propositions :

(a) The concepts of the amount of investment and the average period are of no use if they are not to be used in a theory of interest in which there is no fixed capital and interest is not compounded. Yet Jevons used both concepts when discussing fixed capital.

(b) Jevons derived his "general expression" for compound interest, from a consideration of the special, point-input, point-output, case in which there is an absolute period of production but he then sought to use it, when discussing far more complex cases, by replacing the

[1]It may be noted that taking the wage as given does *not* involve assuming it to be at a biological subsistence level, as Ricardo himself pointed out (*Principles*, pp. 96–97). Nor does Ricardo's approach involve ignoring the existence of different grades of labour, since the wage of each grade may be taken as given. Jevons's criticism of Ricardo's treatment of wages would appear to miss both these points (see *Theory*, 269–70).

[2]Cf. the following statements of H. S. Foxwell, in his introduction to Menger's "*The Right to the Whole Produce of Labour*" (pp. xi–xii) ;

"... it was Ricardo's crude generalisations which gave modern socialism its fancied scientific basis, and provoked, if they did not justify, its revolutionary form".

"... Ricardo ... did more than any intentionally socialist writer to sap the foundations of that form of society which he was trying to explain. ..."

[3]See Jevons's discussion of wages theory in the Preface to the second edition of the *Theory*.

[4]As the careful reader will have already noticed, we have done no more than assert the *consistency* of a number of aspects of Jevons's thought ; we have neither stated nor suggested that one of Jevons's views, beliefs or theories caused him to adopt his other views, beliefs and theories.

absolute period by an average period. This can be done neither for the average period which Jevons defined explicitly, nor for any other average period defined independently of the rate of interest.

(c) Jevons asserted the existence of an inverse relation between average period and interest rate, of a positive relation between value of capital per man and average period and of an inverse relation between value of capital per man and the interest rate. He did not and could not provide a rational basis for any of these three assertions. Furthermore, Jevons attempted to "close" his system by taking as given the value of capital per man but he provided no rationale for this procedure and it is not clear that any rationale can be given for it.

(d) It follows from the propositions asserted in (c) that Jevons's theory failed to determine the rate of interest. It follows, in turn, that Jevons's theory determined nothing ; the interest rate, the wage rate, all prices and all quantities are left undetermined.

As is well-known, both Böhm-Bawerk and Wicksell developed Jevons's theory of interest, based on the concept of a period of production. Neither of them, however, succeeded in removing the central flaws of Jevons's analysis.[1] Jevons wrote[2] that "that able but wrong-headed man, David Ricardo, shunted the car of Economic science on to a wrong line" ; one could, with at least equal justice, say the same of William Stanley Jevons.

IAN STEEDMAN

University of Manchester.

[1]We cannot, of course, establish the truth of this assertion here ; see, instead, Garegnani, *Il Capitale, ecc.*, Part II.

[2]In the final paragraph of his Preface to the second edition of the *Theory*.

WORKS CITED

R. D. C. Black, *Editorial Introduction,* to W. S. Jevons, *Theory of Political Economy,* Penguin edition, London, 1970.

H. S. Foxwell, *Editorial Introduction* to A. Menger, *The Right to the Whole Produce of Labour,* London, 1899.

P. Garegnani, *Il Capitale nelle Teorie della Distribuzione*, Milano, 1960.

W. S. Jevons, *The Importance of Diffusing a Knowledge of Political Economy*, Manchester, 1866.

W. S. Jevons, *Investigations in Currency and Finance*, edited by H. S. Foxwell, London, 1884.

W. S. Jevons, *Letters and Journal of W. Stanley Jevons*, edited by H. A. Jevons, London, 1886.

W. S. Jevons, *Methods of Social Reform, and other papers*, London, 1883 and 1904.

W. S. Jevons, *Primer of Political Economy*, London, 1878.

W. S. Jevons, *The Theory of Political Economy*, 4th Edition, London, 1911 and 5th edition, New York, 1965.

W. S. Jevons, *The State in Relation to Labour*, London, 1882.

J. M. Keynes, "William Stanley Jevons, 1835–1882", *Journal of the Royal Statistical Society*, 1936, pp. 516–48.

D. Ricardo, *Principles of Political Economy*, Cambridge, 1966.

L. Robbins, "The Place of Jevons in the History of Economic Thought", *The Manchester School*, 1936, pp. 1–17.

J. A. Schumpeter, *History of Economic Analysis*, London, 1954.

G. J. Stigler, *Production and Distribution Theories, The Formative Period*, New York, 1948.

K. Wicksell, *Lectures on Political Economy*, London, 1946.

K. Wicksell, *Selected Papers in Economic Theory*, New York, 1969.

K. Wicksell, *Value, Capital and Rent*, London, 1954.

P. H. Wicksteed, *The Commonsense of Political Economy*, London, 1933.

[25]

POSITIVE PROFITS WITH NEGATIVE SURPLUS VALUE[1]

In an excellent paper (1973) Wolfstetter has demonstrated the truth of the proposition that the existence of positive surplus value, as defined by Marx, is a necessary and sufficient condition for the existence of positive prices yielding positive profits. His demonstration is based upon a number of assumptions common in economic theory, including the assumption that there is no joint production. Morishima (1973), who has also proved this proposition, has described it as the "Fundamental Marxian Theorem". We shall neither challenge Wolfstetter's arguments nor discuss Morishima's suggestion that the proposition in question is fundamental to Marxist theory; we shall, however, seek to show that if we make all Wolfstetter's assumptions *except* that we allow joint production, then the proposition is false.[2] In brief, in the presence of joint production, the existence of positive surplus value is neither a necessary nor a sufficient condition for the existence of positive profits and prices. Our procedure will be to provide simple numerical counter-examples to the proposition; in the text we confine our attention to the more important necessity part, our counter-example to the sufficiency part being relegated to a footnote.

The Assumptions

We shall consider a capitalist economy undergoing steady growth, without technical progress, in which workers save nothing, while capitalists have a savings ratio of unity. There are two additive, divisible, constant-returns-to-scale processes of production, which have the same production period; this period will be taken as the time unit for our analysis. There are two commodities, both of which, together with homogeneous labour, serve as inputs to production and are fully used up in one period. We assume, that is, that capital is circulating capital, fixed capital being absent. Both processes, however, are joint-production processes, producing positive quantities of each commodity. Exchange of commodities takes place, at the end of each period, on fully competitive markets. Wages are paid at the end of the period; they do not depend on the supply of labour, which is always at least equal to the demand. Table I shows the commodity inputs to and outputs obtained from each process when it is operated by one unit of labour; inputs being shown to the left of the arrows and outputs to the right.

[1] So many people have made helpful comments on earlier versions of this paper that it would be impracticable to name them all here; their help is gratefully acknowledged.

[2] While we shall refer explicitly only to the work of Morishima (1973; 1974), Sraffa (1960) and Wolfstetter (1973), it is proper to note here that we have been influenced by the work of Gilibert unpublished) and Schefold (unpublished).

The Price System

We assume that the real wage bundle contains, for every 6 units of labour, 3 units of the first commodity and 5 of the second. This given real wage, together with the technical conditions, will determine the profit rate and commodity prices. Given our assumptions about savings behaviour, the growth rate will also be determined, being equal to the profit rate.

Let the labour commanded by a unit of the first (second) commodity be p_1 (p_2) and the uniform profit rate be r. Then from Table I we see that the following relations must hold:

$$(1+r)\,5p_1+1 = 6p_1+p_2 \quad . \quad . \quad . \quad (1)$$

$$(1+r)\,10p_2+1 = 3p_1+12p_2 \quad . \quad . \quad . \quad (2)$$

Furthermore, the real wage bundle which is purchased by 6 units of labour must command 6 units of labour, so that we have:

$$3p_1+5p_2 = 6 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As the reader may easily check by substitution, the solutions to (1), (2) and (3) are:[1]

$$r = +20\%, \quad p_1 = \tfrac{1}{3}, \quad p_2 = 1$$

Note that the profit rate and the prices are all positive.

TABLE I

	Commodity 1		Commodity 2		Labour		Commodity 1		Commodity 2
Process 1	5	+	0	+	1	→	6	+	1
Process 2	0	+	10	+	1	→	3	+	12

The Quantity System

Suppose that, in a certain period, 6 units of labour are employed, 5 operating the first process and 1 operating the second. The resulting input and output flows for this period will be as shown in Table II, where the third row shows the economy-wide situation resulting from the simultaneous operation of the two processes. This situation is consistent with steady growth, at a rate equal to the profit rate. To see this, consider Tables II and III together. Total input is (25+10) and net investment is (5+2); clearly

TABLE II

	Commodity 1		Commodity 2		Labour		Commodity 1		Commodity 2
Process 1	25	+	0	+	5	→	30	+	5
Process 2	0	+	10	+	1	→	3	+	12
Total	25	+	10	+	6	→	33	+	17

[1] As the reader may also check by substitution, there is an alternative solution, namely $r = 1{\cdot}\dot{6}\%$, $p_1 = -12/19$, $p_2 = 30/19$. The negative price p_1 makes this solution economically insignificant.

$(5+2) = 20\,\%\,(25+10)$. Thus net investment is just that required for steady growth at a rate equal to the profit rate.

From the price and quantity systems, we see that all inputs and outputs, prices, wages and growth and profit rates are positive. Hence nothing abnormal would be observed in our economy.

Table III

	Commodity 1		Commodity 2
Net product	8	+	7
Wage bundle	3	+	5
Net investment	5	+	2

The Value System

We now derive the value of each commodity, *i.e.* the quantity of labour required, directly and indirectly, to produce each unit of net output of that commodity. With joint production one cannot follow the method of reduction to dated labour (Sraffa, 1960, chap. VI) for one cannot directly allocate the labour input to a process between the two outputs. Instead, one must have recourse to a simultaneous determination of values. Let the value of the first (second) commodity be l_1 (l_2). Then we see from Table I that

$$5l_1+1 = 6l_1+l_2 \qquad . \quad . \quad . \quad . \quad (4)$$

$$10l_2+1 = 3l_1+12l_2 \qquad . \quad . \quad . \quad . \quad (5)$$

As the reader may easily check by substitution, the solution to (4) and (5) is:[1]

$$l_1 = -1, \quad l_2 = 2$$

We can now calculate V, the "value of labour power", that is the labour embodied in the wage bundle, and the "surplus value" S, which is the labour embodied in the bundle of commodities (net of replacement) appropriated by the capitalists. If our calculations are correct we must, of course, find that $(V+S) = 6$ = total labour employed. Now the bundle of commodities appropriated by the workers is $(3+5)$, while that appropriated by the capitalists is $(5+2)$. Hence

$$\begin{aligned} V &= 3\times(-1)+5\times 2 = 7 \\ S &= 5\times(-1)+2\times 2 = -1 \\ V+S &= 6 \end{aligned}$$

Thus we find that surplus value is negative ($S = -1$), while the rate of profit is positive ($r = 20\,\%$). In a footnote we provide a numerical example in which surplus value is positive, while the growth and profit rates are

[1] Note that $(l_2/l_1) = -2$ while $(p_2/p_1) = 3$. Thus relative price can diverge from relative value not only in magnitude but also in sign.

negative.[1] We conclude that, with joint production, the existence of positive surplus value is neither a necessary nor a sufficient condition for the existence of positive profits.[2]

We have not argued, it should be noted carefully, that with joint production surplus value and profit must be of opposite sign; we have merely shown that they can be. Returning to the section "The Price System", it will be clear that any real wage bundle $(w_1 + w_2)$ which satisfies the relation

$$(\tfrac{1}{3})w_1 + w_2 = 6 \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

will be consistent with the solution $r = +20\%$, $p_1 = \frac{1}{3}$, $p_2 = 1$. The corresponding values of S and V will be given by

$$V = -w_1 + 2w_2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$S = 6 - V = 6 + w_1 - 2w_2 \qquad . \quad . \quad . \quad . \quad (8)$$

From (6), (7), (8) we see that

$$V = 5w_2 - 18$$

and

$$S = 24 - 5w_2$$

where $0 \leqslant w_2 \leqslant 6$. The *rate* of surplus value, or rate of exploitation, e, is defined by $e \equiv S/V$ so that

$$e = (24 - 5w_2)/(5w_2 - 18)$$

for $0 \leqslant w_2 \leqslant 6$. It is easy to show that as we notionally vary w_2 between 0 and 6, e takes on every value, from minus to plus infinity, except that it never

[1] To provide a counter-example to the assertion that positive surplus value is a sufficient condition for positive profit, we must construct a case in which surplus value is positive but profits are negative. Since capitalists could, in fact, choose to obtain a zero profit rather than a negative one, simply by ceasing to be capitalists, it will be clear that the example is of a purely formal nature. Consider then an economy with the same available processes and the same savings behaviour as assumed in the text. Let the real wage bundle be (20·5 + 12) for every 6 units of labour. Equations (1) and (2) of the text will still apply but (3) must be replaced by

$$20{\cdot}5p_1 + 12p_2 = 6 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3')$$

As the reader can easily check by substitution, the solution to (1), (2) and (3′) is

$$r = -50\%, \quad p_1 = 12/43, \quad p_2 = 1/43$$

(There is also an economically irrelevant solution, namely that given in note 1, p. 115.) It can also be seen that the quantity system given in Table II of the text is compatible with steady growth at the rate of -50%, for the net output of (8 + 7) *minus* the wage bundle of (20·5 + 12) leaves [(−12·5) + (−5)] for net investment and the latter is clearly -50% (25 + 10) or -50% of the inputs. We can now calculate V and S:

$$\begin{aligned} V &= 20{\cdot}5 \times (-1) + 12 \times 2 &&= 3{\cdot}5 \\ S &= (-12{\cdot}5) \times (-1) + (-5) \times 2 &&= 2{\cdot}5 \\ V + S & &&= 6. \end{aligned}$$

Thus we have positive surplus value ($S = 2{\cdot}5$) but negative profits ($r = -50\%$) with positive prices. Hence positive surplus value is not a sufficient condition for positive profits.

[2] In his recent inaugural lecture (1974) Morishima has shown that "Positive exploitation is necessary and sufficient. . . to guarantee capitalists positive profits." This result may appear to be in stark contrast with ours but there is, in fact, no inconsistency between the two results, since Morishima adopts definitions of surplus and exploitation quite different to those used above. Morishima, it could be argued, obtains a *theorem* with a distinctly Marxist flavour, by abandoning the traditional Marxist *analysis*.

falls in the open interval $(-\frac{4}{3}, -\frac{1}{2})$. Hence ***almost every*** value of the rate of exploitation, positive or negative, is consistent with $r = +20\%$.

While we are considering the value system, it may be of interest to convert Table II to a set of "value accounts" in which inputs and outputs are "valued" in terms of embodied labour. As is usual in Marxist analysis, we use C to denote labour embodied in the produced means of production; we use E to denote the value of gross output. Then if we apply the results obtained above $[l_1 = -1,\ l_2 = 2,\ V = 7,\ S = (-1)]$, we obtain Table IV from Table II. In addition to surplus value being negative for each process and in total, we see that both C, the "value of constant capital", and E are negative for process one, while C is negative for the economy as a whole; in total, the produced means of production embody a negative amount of labour. Further the "organic composition of capital", *i.e.* C/V, is negative for process one and for the economy as a whole.

Table IV

	C		V		S		E
Process 1	−25	+	35/6	+	(−5/6)	=	(−20)
Process 2	20	+	7/6	+	(−1/6)	=	21
Total	−5	+	7	+	(−1)	=	1

Discussion

Since the above numerical results may appear somewhat strange, it may be helpful to provide an intuitive interpretation of negative value and negative surplus value. One reason for finding "negative value" rather odd, is that one is used to thinking of the Marxian value of a commodity as the labour required to produce a net output consisting of that commodity alone. Reflection will show, however, that with joint production it is, in general, *not possible* to produce only one commodity, this being so even if all values are positive. Consider Fig. 1, where A, B are the net output points for two different processes. The downward slope of AB shows that both commodities have positive value. By appropriately allocating labour between the processes, net outputs lying on AB can be produced but points C and D, with only one commodity being produced, cannot be reached. A purely *formal* solution exists, it is true; C can be reached by allocating a *positive* amount of labour to process OA and a *negative* amount of labour to process OB, the difference between these two amounts of labour being the value of OC units of commodity 2. (A parallel argument holds for commodity 1.) Even though each commodity has a positive value, it is not sensible to think of these values as quantities of labour required to produce only the given commodity, since such production involves (meaningless) negative employment in one process. (If a commodity has a negative value, as in our example, this simply means that the hypothetical negative employment in one process outweighs the positive employment in the other process.) Once it is seen that, with joint production, one should not conceive of the value of a commodity as the labour required to produce that commodity alone, the "oddness" of a negative value disappears.

The more appropriate way to conceive of value is as the *change in employment* resulting from a change in net output from (y_1, y_2) to (y_1+1, y_2) or (y_1, y_2+1), where each output can be produced by some meaningful, positive

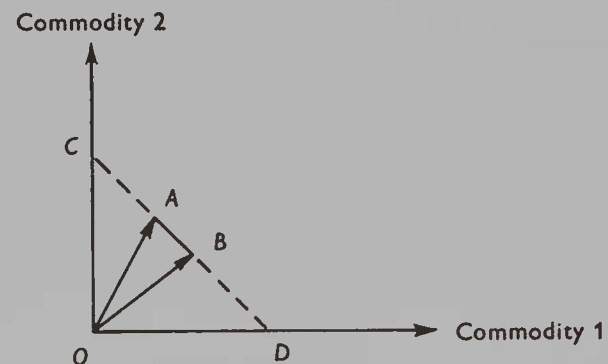


Fig. 1

allocation of labour between the processes.[1] We have seen above that our economy, growing at 20 % and paying a wage of (3 + 5) to 6 units of labour, produces a net output—that is, wages plus net investment—of (8 + 7). Now consider another economy, with the same methods of production and also growing at 20 %, and suppose that the real wage paid to 5 units of labour is (6 + 3). It is easy to show that net output in this economy will be (9 + 7). Thus, by comparison with our first economy, the second has the same net output of commodity 2, yet produces one *more* unit of commodity 1 even though employment is one unit *smaller*. This is the meaning of the result that commodity 1 has a value of − 1. Again, consider a third economy, with the same methods of production and growing at 20 %, which pays a real wage of (3 + 6) to 7 units of labour; the net output will be (9 + 8). By comparison with the second economy, the third produces the same net output of commodity 1 but one *more* unit of commodity 2 and employs two *more* units of labour. This is the meaning of the result that commodity 2 has a value of + 2. Thus, if value is conceived of in a way which has meaning in the context of joint production, there is nothing at all strange about a negative value. It follows that there is nothing strange about negative surplus value.

In Fig. 2 point I_1 (I_2) shows the input pattern that would obtain if 6 units of labour were allocated to process 1 (2), while point N_1 (N_2) shows the corresponding net output pattern. Naturally, all the points on the straight line $N_1 N_2$ are possible net output patterns with total employment equal to 6. The fact that $N_1 N_2$ has a positive slope shows immediately that commodities one and two have values of opposite sign. The points I and N show the input and net output patterns displayed in Table II.

[1] Cf. Sraffa (1960), p. 60, second paragraph, and Morishima (1973), p. 18: "...values are not more than the employment multipliers discussed by Kahn and later by Keynes...".

Point W in Fig. 2 shows the wage bundle $(3+5)$. Since W lies above the extension of $N_1 N_2$, it follows that the real wage bundle embodies more than 6 units of labour and hence that surplus value is negative.

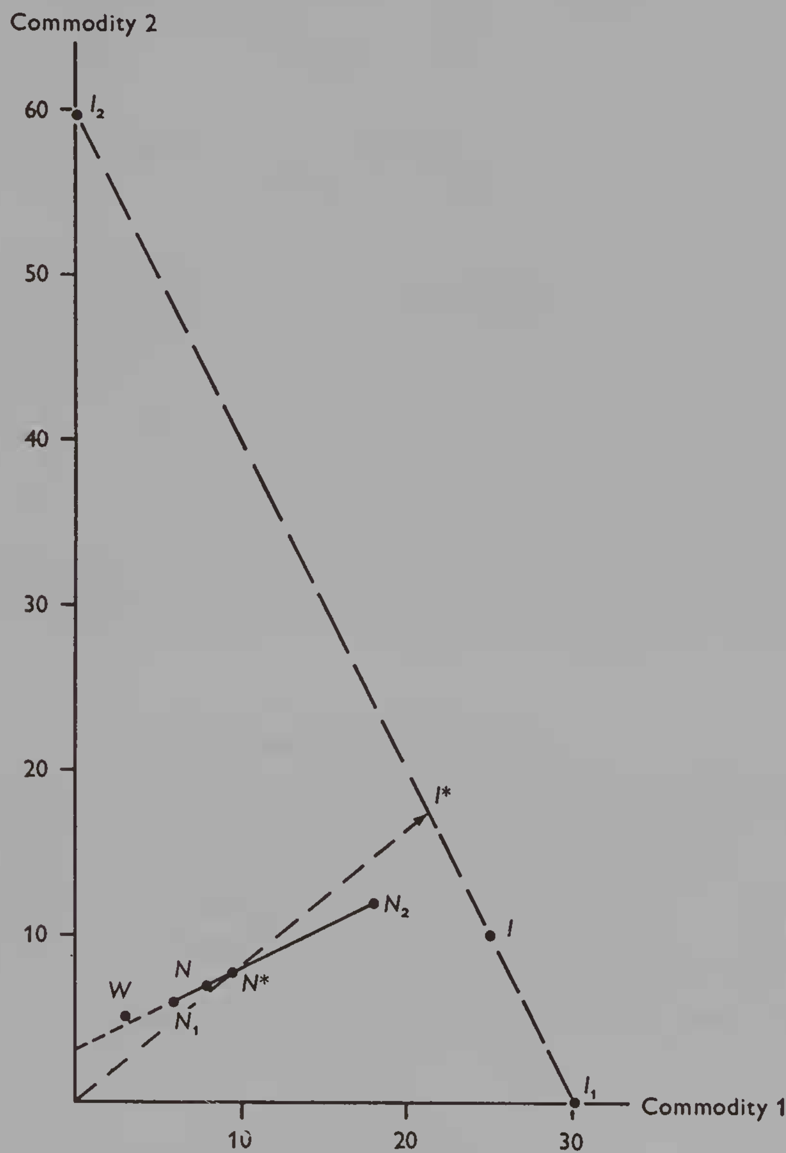


Fig. 2

An Implicit Assumption

It was taken for granted in the argument above that both processes were operated at a positive level. Since, however, the first process produces a net output of only $(1+1)$ when operated with one unit of labour, while the second process, with the same employment, produces a net output of $(3+2)$, it might be questioned whether the first process will be used. We must show that it will.

Returning to Table I, if process 1 is operated by 6 units of labour which are paid a real wage of $(3+5)$, the condition that super-normal profits, over and above the normal profits at rate r, should be negative or zero can be written as

$$30p_1(1+r)+3p_1+5p_2 \geqslant 36p_1+6p_2$$

or

$$30p_1 r \geqslant 3p_1+p_2 \quad . \quad . \quad . \quad . \quad (9)$$

In the same way, we have for process 2,

$$60p_2(1+r)+3p_1+5p_2 \geqslant 18p_1+72p_2$$

or

$$60p_2 r \geqslant 15p_1+7p_2 \quad . \quad . \quad . \quad . \quad (10)$$

Now p_1 and p_2 cannot both be zero; we therefore see from (9) that p_1 cannot be zero and from (10) that p_2 cannot be zero. Hence, in a competitive equilibrium, both p_1 and p_2 must be positive and therefore excess demand for each commodity must be exactly zero. Each process, however, produces a positive supply of both commodities, even after the wage bundle has been subtracted, but uses only one commodity as an input. Thus, given our assumptions about savings, it is impossible for either process alone to generate a zero excess demand for each commodity. Hence if a competitive equilibrium exists, it *must* involve the operation of both processes. We have, of course, already given an example of such an equilibrium in our sections "The Price System" and "The Quantity System".[1]

The Standard Commodity

Some readers may wonder whether our "odd" results are connected with "strange" behaviour of the Sraffa Standard Commodity (1960, chaps. IV, VIII), such as the latter containing negative quantities. In fact the Standard Commodity for our example is perfectly "normal". Taking total employment to be unity now, allocate $[\sqrt{6}/(1+\sqrt{6})]$ units of labour to the first process and $[1/(1+\sqrt{6})]$ to the second. The resulting pattern of inputs and outputs is shown in Table V, where $D = (1+\sqrt{6})$. It will be seen that

$$\begin{aligned} \textit{Net}\text{ Product} &= [(3+\sqrt{6})/D+(2+\sqrt{6})/D] \\ &= \left(\frac{2+\sqrt{6}}{10}\right)[5\sqrt{6}/D+10/D] \\ &= \left(\frac{2+\sqrt{6}}{10}\right)[\text{gross input}] \end{aligned}$$

The commodities enter net product and gross input in the same proportions and thus, by definition, this net product is the Sraffa "standard net product".

[1] It may be noted that positive growth is not essential for our conclusion that negative surplus value is consistent with positive profits and prices. Suppose, for example, that capitalists save nothing and consume the commodities in the fixed proportions of five units of commodity 1 for every two units of commodity 2. A competitive equilibrium will exist with exactly the same prices, quantities and values as given in the text.

TABLE V

	Commodity 1		Commodity 2		Labour		Commodity 1		Commodity 2
Process 1	$5\sqrt{6}/D$	+	0	+	$\sqrt{6}/D$	→	$6\sqrt{6}/D$	+	$\sqrt{6}/D$
Process 2	0	+	$10/D$	+	$1/D$	→	$3/D$	+	$12/D$
Total	$5\sqrt{6}/D$	+	$10/D$	+	1	→	$(3+6\sqrt{6})/D$	+	$(12+\sqrt{6})/D$

It is perfectly "normal", being real and strictly positive.[1] Furthermore, the maximum possible rate of profit, R, is real and positive, being given by

$$R = 10(2+\sqrt{6})\,\% = 44{\cdot}5\,\%$$

Now let W, P_1, P_2 be the wage, the price of the first and the price of the second commodity respectively in terms of the standard net product. It is easy to show that

$$W = [1-(5\sqrt{6}-10)r] \quad . \quad . \quad . \quad (11)$$

$$P_1 = (10r-1)/[(10+5\sqrt{6})r+1] \quad . \quad . \quad . \quad (12)$$

and

$$P_2 = (5r+2)/[(10+5\sqrt{6})r+1] \quad . \quad . \quad . \quad (13)$$

The graphs of (11), (12) and (13) for $0 \leqslant r \leqslant R$ are shown in Fig. 3. It will be seen that P_1 rises monotonically with r, starting at $P_1(0) = -1 = l_1$, and is positive only for $r > 10\,\%$.[2] P_2 falls monotonically with r, starting at

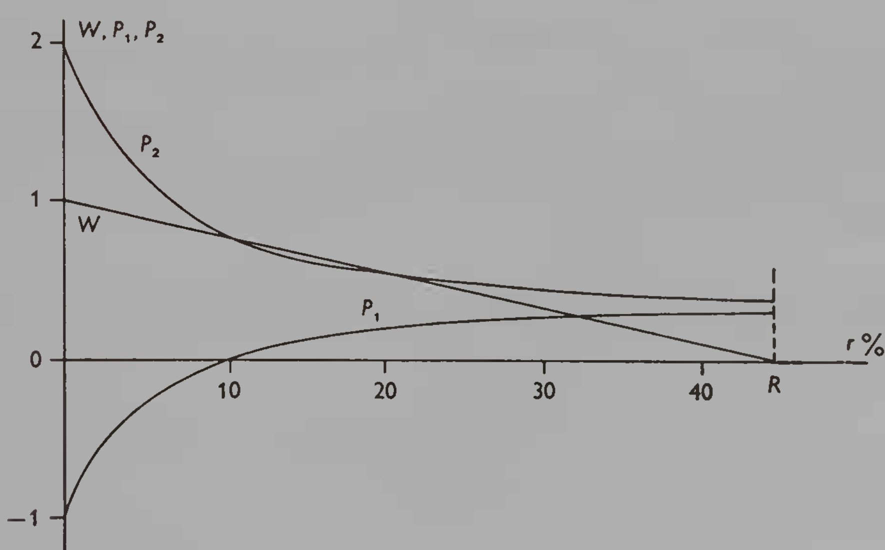


FIG. 3

$P_2(0) = 2 = l_2$, and is equal to W both at $r = 10\,\%$ and $r = 20\,\%$. (W/P_2), the wage in terms of the second commodity, at first rises with r, then reaches a maximum[3] and thereafter falls with r. (W/P_1), the wage in terms of the first commodity, always falls as r rises but is positive only for $r > 10\,\%$. At

[1] In Fig. 2, points I^* and N^* show the inputs and net outputs for a Standard System employing 6 units of labour.

[2] $P_1 = W$ when $r = 10(1+\sqrt{5})\,\% = 32{\cdot}36\,\%$.

[3] At $r = 10(\sqrt{30}-4)\,\% = 14{\cdot}77\,\%$.

$r = 20\%$, $W = (3-\sqrt{6})$, $P_1 = 1/(3+\sqrt{6})$, $P_2 = 3/(3+\sqrt{6})$; it is easy to check that these results are consistent with the $p_1 = 1/3$, $p_2 = 1$ found above.

One can construct any number of hypothetical economies having the above two-commodity system as their basic system and having, in addition, many single product processes producing non-basic commodities, each of which has a negative labour value but a positive price.[1]

Conclusion

We have shown that with joint production, the existence of positive surplus value is neither a necessary nor a sufficient condition for the existence of positive profits. *If* either of these results is important for Marxist economics it is presumably the first (lack of necessity) rather than the second (lack of sufficiency), since economies with a negative profit rate are, we assume, of only formal and not real interest.

Ian Steedman

University of Manchester.

Date of receipt of final typescript: September 1974.

References

Gilibert, G. (Unpublished). *Produzione Congiunta e Valori-Lavoro Negativi.*

Morishima, M. (1973). *Marx's Economics. A Dual Theory of Value and Growth.* Cambridge University Press.

Morishima, M. (1974). "Marx in the Light of Modern Economic Theory." *Econometrica*, pp. 611–632.

Schefold, B. (Unpublished). *Mr. Sraffa on Joint Production.*

Sraffa, P. (1960). *Production of Commodities by Means of Commodities.* Cambridge University Press.

Wolfstetter, E. (1973). "Surplus Labour, Synchronised Labour Costs and Marx's Labour Theory of Value," Economic Journal, Vol. 83 (September), pp. 787–809.

[1] Thus suppose that a single-product process i uses a_{i1} units of 1, a_{i2} units of 2 and one unit of labour to produce one unit of commodity i. The value of, or labour embodied in, one unit of i, l_i, will be given by

$$l_i = -a_{i1} + 2a_{i2} + 1$$

so that $l_i < 0$ provided that $a_{i1} > (2a_{i2}+1)$. The standard commodity price of i, P_i, will be given by

$$P_i = (1+r)\ (a_{i1}P_1 + a_{i2}P_2) + W$$

and this will certainly be positive for $10\% \leqslant r \leqslant R$, whether or not l_i is negative.